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MEMORANDUM

To: Bobby Johnston

From: Peter Fisher

Subject: Transmission lines

Date: August 28, 2018

A transmission line consists of two conductors in close proximity to each other, usually a fixed distance apart, shown conceptually in Fig. **??**. A transmission line carries high frequency signals between two places separated by more than a wavelength. The signal's energy propagates through the space between the conductors, usually through a dielectric. The standard RG-58 co-axial transmission lines provides good example, Fig. **??**. The memo provides an analysis of co-axial transmission lines that generalizes to other types of transmission lines.

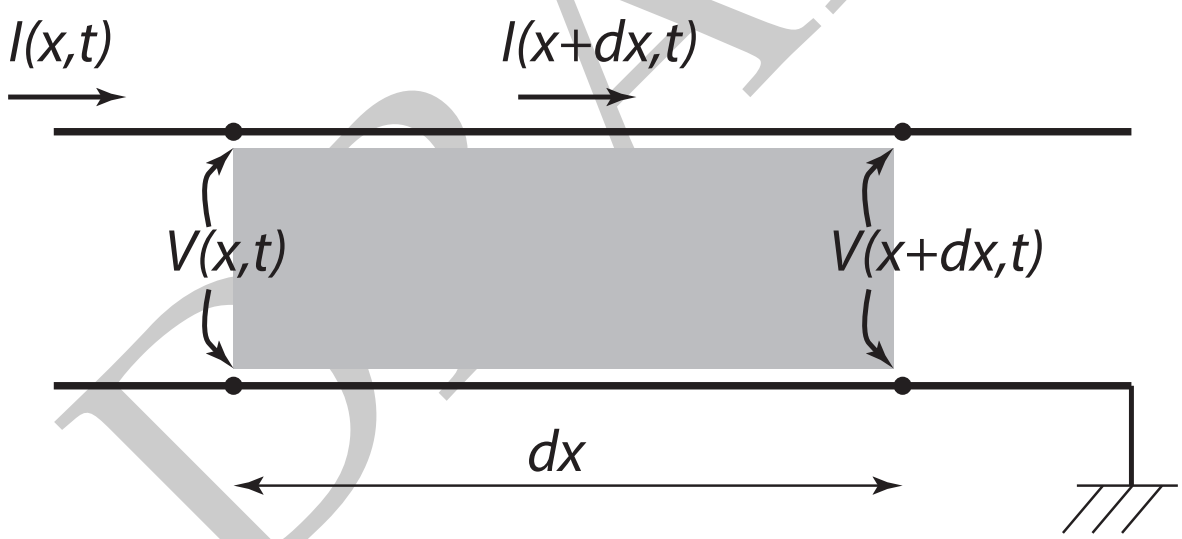


Figure 1: Segment of length dx of conceptual transmission line with two conductors. The grey rectangle is the region described in the text for the Faraday's law calculation.

1 The General Idea of a Transmission Line

The transmission line supports a time and position varying voltage and current between the conductors. One conductor may be grounded, in which there will be a linear charge density $\lambda(x,t)$ and current $I(x,t)$ on the ungrounded conductor. In a little segment along the conductor from x

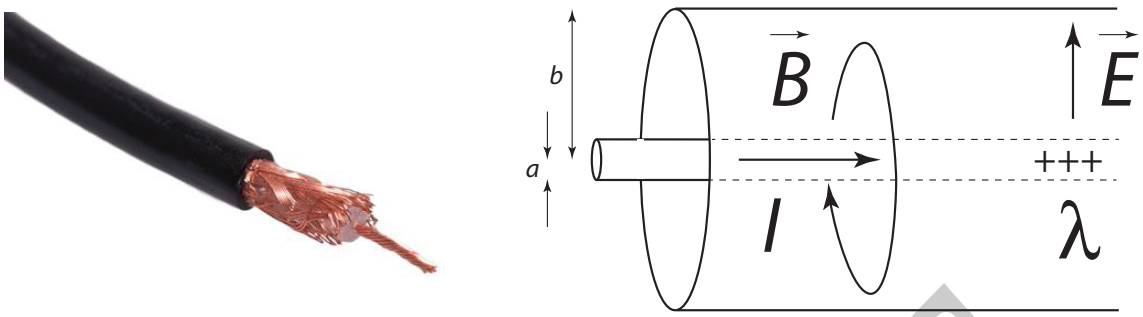


Figure 2: Left: picture of the end of RG-58 coaxial cable. The inner conductor is copper, surrounded by white insulator, in turn surrounded by copper braid, which is grounded. Black insulation surrounds the copper braid. Right: Drawing of ideal RG-58.

to $x+dx$, the current flowing into the segment is $I(x,t)$, out is $I(x+dx,t)$, so change conservation says the rate of charge accumulation in the segment is,

$$\frac{dQ}{dt} = \dot{\lambda}(x,t) = I(x+dx,t) - I(x,t) = \frac{dI}{dx}dx.$$

$Q = \lambda dx$ There is a capacitance $C = \mathcal{C}dx$ between the conductors so the potential at x is

$$V = \frac{Q}{C} = \frac{\lambda}{\mathcal{C}}$$

where $\mathcal{C}$ is the capacitance per unit length. Then $\dot{V} = \dot{\lambda}/\mathcal{C}$ and we can combine the two equations to get,

$$\mathcal{C}\frac{dV}{dT} = \frac{dI}{dx}. \tag{1}$$

A loop along the top conductor from x to $x+dx$, between the conductors the point $x+dx$, back along the bottom conductor from x, and finally back to the starting point from bottom to top at x, (grey rectangle in Fig. **??**) will have a magnetic flux through it from the current flowing in the top conductor. Faraday's law says,

$$\int_{\text{Area}} \vec{B}\cdot d\vec{a} = \int_{\text{Surface}} \vec{E}\cdot d\vec{l} \rightarrow \mathcal{E} = L\frac{dI}{dt},$$

where $\mathcal{E}$ is the electromotive force and L is the inductance. In this case,

$$\mathcal{E} = V(x+dx,t) - V(x,t) = \mathcal{L}dx\frac{dI}{dt}$$

where $\mathcal{L} = L/dx$ is the inductance per unit length. Then,

$$\frac{V(x+dx,t) - V(x,t)}{dx} = \frac{dV}{dx} = \mathcal{L}\frac{dI}{dt}. \tag{2}$$

Differentiate Eq. **??** with respect to x, Eq. **??** with respect to t and eliminate the $d^2V/dxdt$ between the equations to get,

$$\frac{d^2I}{dt^2} = -\frac{1}{\mathcal{LC}}\frac{d^2I}{dx^2},$$

that describes a current wave propagating with velocity $v = 1/\sqrt{\mathcal{LC}}$ down the upper conductor. There is an identical expression for V and using Eq. **??**,

$$V(x,t) = V_o f(x \pm vt) = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} I_o f(x \pm vt),$$

and $Z = \sqrt{\mathcal{L}/\mathcal{C}} = V_o/I_o$ is the impedance of the transmission line. Since Z is the ratio of voltage to current, it has the units of Ohms. Also,

$$Z = \frac{\mathcal{L}}{\mathcal{C}} = \frac{V}{I} = \frac{\lambda}{2\pi\epsilon I}\ln\frac{b}{a} = \frac{\lambda}{I}\frac{1}{\mathcal{C}} = \frac{\lambda}{I} \rightarrow I = v\lambda.$$

As the current pulse propagates down the transmission line, the is a linear charge density pulse along with it. The solutions are $I(x,t) = I_o f(x \pm vt)$ where the negative sign is for waves propagating toward $+x$ and the negative is for waves propagating in $-x$.

The simplest wave is a step wave,

$$I(x,t) = I_o \theta(x \pm vt),$$

where $\theta(u) = 1$ if $u > 0$ and 0 otherwise and is known as the Heavyside function. Fig. **??** shows a step wave propagating in the $+x$ direction. Ahead of the wave, $E = B = 0$ and the are no charges or currents. Behind the wave, a current I_o flows and the analysis above shows $V_o = I_o Z$, so there is a linear charge density on the upper conductor.

2 Terminating a transmission line

What happens when the wave reaches the end of the transmission line? Suppose a resistor R connects the two conductors, Fig **??**. The arriving step will apply a voltage V_o across the resistor, causing a current $I_R = V_o/R$ to flow. A current pulse $I_o = V_o/Z$ arrives at the same time as the voltage step. If $R = Z$, then $I_R = I_o$ and all the current accompanying the voltage step flows into the resistor, where it is dissipated as heat. If $I_o > I_R$, then the additional current has to go somewhere. The additional current, equal to $I_o - I_R$ cannot flow into the resistor as the relation $V_o = RI_R$ must be maintained. The additional current can only flow on the transmission line if it obeys $I(x,t) \propto f(x \pm vt)$ and the current is at the end of the line, so the additional current is reflected and goes back down the line: $I_r = I_{o,r}(x + vt)$ and $V_{o,r} = I_{o,r}Z$. The total voltage along the line behind the reflected pulse is $V_o + V_{o,r}$ and in front of the pulse, it is just V_o. In front of the pulse, the current is still $I_o = V_o/Z$, but behind the pulse, the reflected current I_r flowing in the $-x$ direction partially cancels the current I_o flowing in the $+x$ direction. The magnetic field induced by the currents is similarly effected.

If $I_R > I_o$, more current has to flow out of the transmission line into the resistor to maintain the current-voltage relationship. This current can only come from a step function wave of negative current propagating in the $-x$ direction, $I_r = I_o - I_R$. Since the net current is less than zero, the voltage pulse must also be negative.

This results in three important cases:

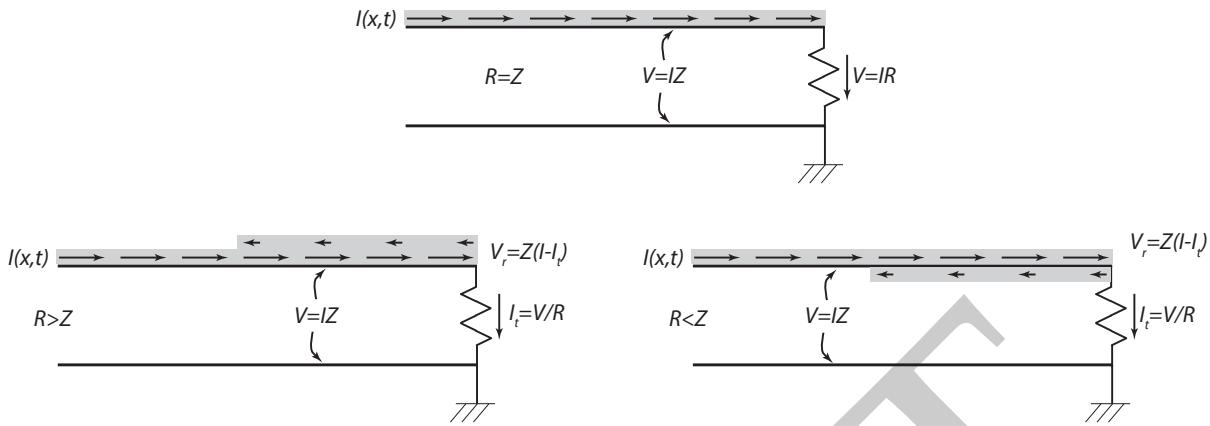


Figure 3: Top graph shows $R = Z$ termination for which all the current goes through the resistor. Left and right show $R > Z$ and $R < Z$, respectively. The grey regions show the direction of current and size of voltage in the reflected wave.

1. $R = Z$ - in this case, all the current flows into R and there are no returning waves.
2. $R = \infty$ - no current can flow into the resistor and all the current reflects back down the line. In front of the pulse the voltage is V_o and current is $I_o = V_o/Z$. Behind the step, the voltage is $2V_o$ and there is no net current (and no magnetic field.)
3. $R = 0$ - the conductors are shorted, so $V = 0$ and no current can flow between the conductors. A negative wave of $-V_o$ heads back down the line and the reflected wave is $I_r = -I_o\,(x + vt)$. Behind the wave, $V = 0$ and the current $2I_o$.

A transmission line carries a signal from one device to another and generally, the input resistance of the receiving device should be the same, or "matched" to the line impedance.

3 Step function voltage source on a transmission line

Fig. **??** shows an ideal voltage source $\mathcal{E}$ connected to the ungrounded conductor of a transmission line through a resistor R and switch S. S is closed at $t = 0$ and before then, there are no currents or charges in the system. An ideal voltage source will provide whatever current is needed to the load to keep the supply voltage at $\mathcal{E}$. In realty, every voltage source has an internal resistance, in this case R, the models the maximum current that the source can supply. So long as the load resistance is much smaller than R, the voltage after R will be close to $\mathcal{E}$. In the following, $R >> Z$.

When the switch closed, a step voltage moves at $v = c/n$ down the line and, if the length of the line is l, the step will reach the end of the line in time $t_o = l/v$. The ungrounded conductor has no charge on it, so a current $I_o = \mathcal{E}/R$ must flow down the line, with voltage $V_o = I_oZ = Z\mathcal{E}/R$. $R_{end} = \infty$ at the end of the line (there is nothing there), so the pulse is reflected and returns, reaching the near end of the line at $t = 2t_o$. While the pulse traveled down the line, the potential at the near end of the line was V_o. When the return pulse reached the near end of the line, the potential after $t = 2t_o$ is,

$$V_1 = V_o + I_oZ + I_1Z.$$

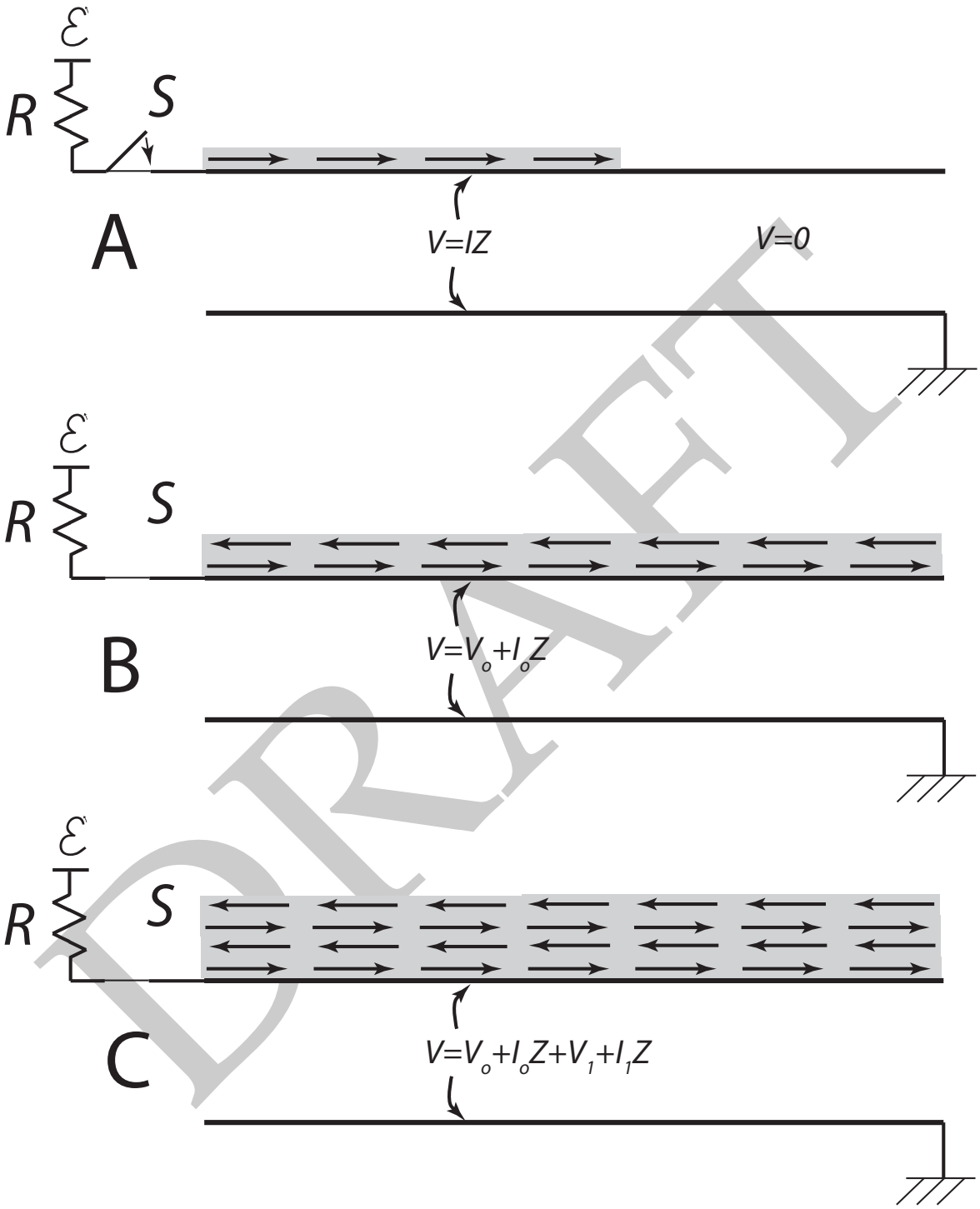


Figure 4: A - current flowing down the line at $t < t_o$. B - return current just as it reaches the near of the of the line at $t = 2t_o$. C - currents just as the second pulse reaches the near end of the line at $t = 4t_o$.

The first V_o is the potential supporting the current going down the line, the second I_oZ is the potential accompanying the return pulse, and I_1Z is the potential for the additional current I_1, which results from the partial reflection of the returning pulse and is equal to,

$$I_1 = \frac{\mathcal{E} - V_1}{R}.$$

The process repeats and at $t = 4t_o$,

$$\begin{aligned} V_2 &= V_1 + I_1 Z + I_2 Z && (3) \\ I_2 &= \frac{\mathcal{E} - V_2}{R}. && (4) \end{aligned}$$

After n round trips, the potential and current are,

$$\begin{aligned} V_n &= V_{n-1} + I_{n-1} Z + I_n Z && (5) \\ I_n &= \frac{\mathcal{E} - V_n}{R}. && (6) \end{aligned}$$

Substituting Eq. **??** into Eq. **??** gives,

$$\begin{aligned} V_n &= V_{n-1} + \frac{(\mathcal{E} - V_{n-1})\, Z}{R} + \frac{(\mathcal{E} - V_n)\, Z}{R} && (7) \\ V_n \left(1 + \frac{Z}{R}\right) &= V_{n-1}\left(1 - \frac{Z}{R}\right) + \frac{2\mathcal{E}Z}{R} && (8) \\ V_n - V_{n-1} &= V_{n-1}\frac{1 - \frac{Z}{R}}{1 + \frac{Z}{R}} - V_{n-1} + \frac{2\mathcal{E}Z}{R} = \frac{dV}{dn} && (9) \\ \frac{dV}{dn} &= -\frac{2Z/R}{1 + Z/R} V + \frac{2\mathcal{E}Z}{R + Z} && (10) \\ &\simeq -\frac{2Z}{R} + \frac{2\mathcal{E}Z}{R}, && (11) \end{aligned}$$

with the last approximation from from the assumption that $R >> Z$. The number of round trips in time t is,

$$n = \frac{t}{t_o} = \frac{t}{\frac{2l}{v}} \rightarrow \frac{dn}{dt} = \frac{v}{2l},$$

and from Eq. **??**,

$$\frac{dV}{dt} = \frac{dV}{dn}\frac{dn}{dt} = -\frac{2Zv}{2l} V + \frac{2\mathcal{E}Zv}{Rl}.$$

$Z = \sqrt{\mathcal{L}/\mathcal{C}}$ and $v = 1/\sqrt{\mathcal{L}\mathcal{C}}$, so $Zv = 1/\mathcal{C}$ and $ZV/l = 1/C$. Finally,

$$\frac{dV}{dt} = -\frac{1}{RC} V + \frac{\mathcal{E}}{RC},$$

which has solution $V(t) = \mathcal{E}\left(1 - \exp -t/RC\right)$.

This shows the cable charges up like a capacitance of $C = \mathcal{C}l$, but the charging happens by a current running back and forth respecting $V = IZ$, laying down a linear charge density λ until the line, or capacitor is fully charged, Fig. **??**. For 400 m line, $t_o \sim 2.5\mu$s, and you can see the steps on an oscilloscope, Fig. **??**.

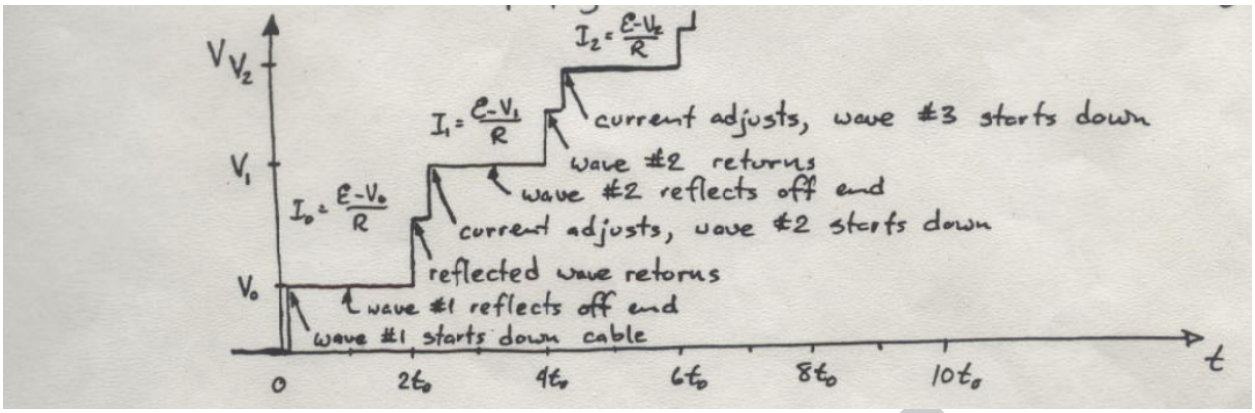


Figure 5: Voltage as a function of time measured at the near end of the transmission line. Source: TSG webpages.

4 The Coaxial Transmission Line

Fig. **??** shows RG-58 cable and Table **??** lists the relevant properties. RG-58 provides a specific geometry that can be used to compute the electric and magnetic fields to find the inductance and capacitance per unit length, which gives the impedance and wave velocity.

Quantity	Value
a	0.41 mm
b	1.42 mm
ϵ	2.25 ϵ_o
v	0.666 c
$\mathcal{C}$	100 pF/m
$\mathcal{L}$	250 nH/m
Z	49.9 Ω

Table 1: Physical characteristics of RG-58 coaxial cable.

The current moving along will create a magnetic field according to Ampere's law,

$$\int_C \vec{B} \cdot d\vec{l} = \mu_o \int_A \vec{J} \cdot d\vec{a}.$$

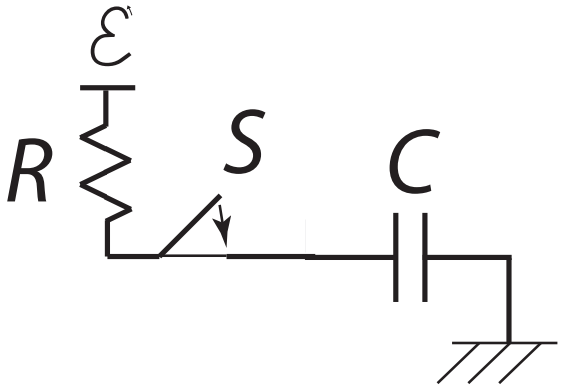
Since the problem is symmetric under rotations around the x-axis, $\vec{B} = B(r)\,\hat{\phi}$,

$$2\pi r B(r) = I\mu_o \rightarrow \vec{B} = \frac{I\mu_o}{2\pi r}\hat{\phi}.$$

Then using Faraday's law to a segment of the cable of length l,

$$\frac{d\Phi}{dt} = L\frac{dI}{dt} =\rightarrow \frac{d}{dt}\int_a^b \frac{\mu_o}{2\pi r} l dr = \frac{\mu_o}{2\pi} = \ln\frac{b}{a} = L\frac{dI}{dt},$$

gives $\mathcal{L} = L/l = (\mu_o/2\pi)\ln b/a$.



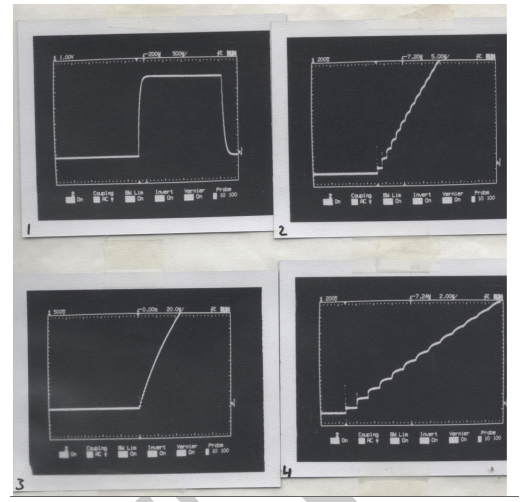


Figure 6: Left: equivalent circuit for transmission line with $C = \mathcal{C}l$. Right: Upper left photo is the voltage pulse sent into the transmission line. The other three photos show various magnifications of the voltage at the end of the transmission line as the wave travels back and forth. Source: TSG webpages

For the capacitance per unit length, recall that in a segment of length l, the total charge will be $Q\lambda l$. The azimuthal symmetry implies that $\vec{E} = E(r)\,\hat{r}$. Gauss law applied to a cylinder of radius r and length l is,

$$\int_A \vec{E} \cdot d\vec{a} = \frac{Q}{4\pi\epsilon} \to E(r) = \frac{Q}{2\pi r \epsilon l}.$$

The potential between the inner and outer conductors is then,

$$V = \int_a^b \vec{E} \cdot d\vec{r} = \frac{Q}{2\pi\epsilon L} \ln \frac{b}{a} = \frac{\lambda}{2\pi\epsilon} \ln \frac{b}{a}.$$

Since $Q = CV$,

$$C = \frac{Q}{V} = \frac{2\pi\epsilon L}{\ln b/a} \to \mathcal{C} = \frac{C}{L} = \frac{2\pi\epsilon}{\ln b/a}.$$

The wave velocity is $v = 1/\sqrt{\mathcal{L}\mathcal{C}} = 1/\sqrt{\epsilon\mu_o} = \frac{c}{n}$, where $n = \sqrt{\epsilon/\epsilon_o}$ is the index of refraction.

The electromagnetic wave traveling down the transmission line carries energy via the Poynting vector,

$$\vec{S} = \frac{1}{\mu_o} \vec{E} \times \vec{B},$$

and the total power carried by the electric and magnetic fields is,

$$P = \int_{\text{Area}} \vec{S} \cdot d\vec{a}$$

which works out, for the fields found above, to $P = V^2/Z$. If the line is terminated with $R = Z$, this is just exactly the power dissipated in the resistor, as it must be.