MEMORANDUM

To: All Concerned

From: Peter Fisher

Subject: Cross section enhancement - classical somerfeld effect

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An object of mass m is incident on and far from an object of mass M with impact parameter b. We know its velocity v and want to find its distance of closest approach δ, Fig. 1. Angular energy

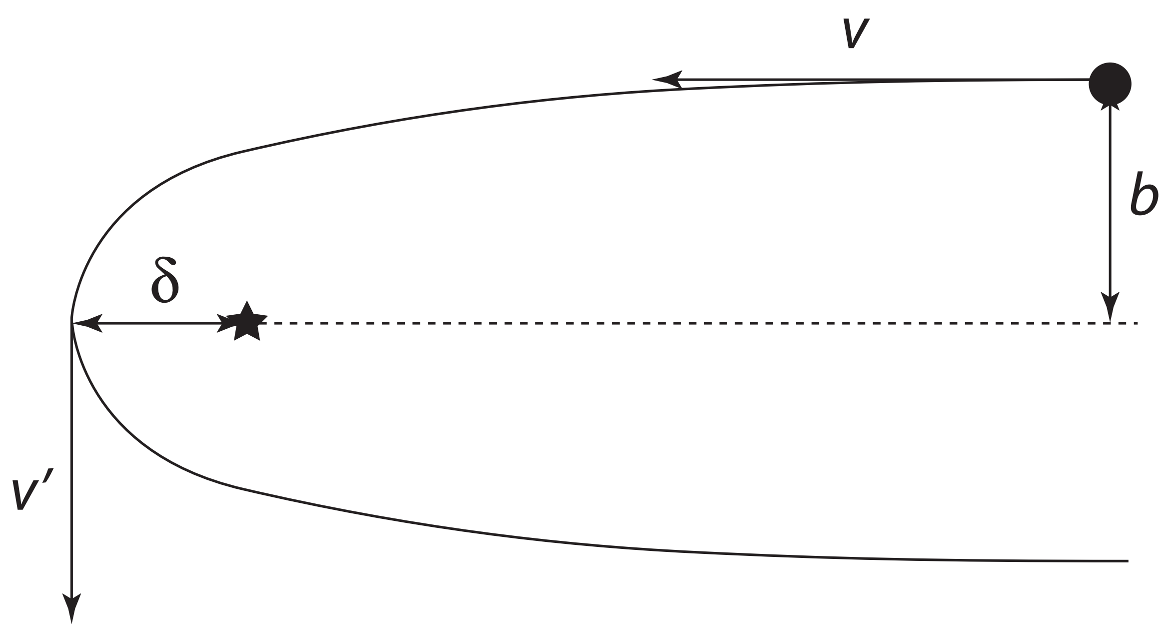

Figure 1: Trajectory for an unbound non-relativistic Kepler orbit.

conservation gives

$$E = \frac{1}{2}mv^2 = \frac{1}{2}mv'^2 - \frac{GMm}{\delta} \quad (1)$$

and energy conservation says,

$$L = mvb = mv'\delta \quad (2)$$

Using

$$r_s = \frac{2GM}{c^2},$$

then, we want to find b in terms of δ. Putting the last three equations together gives,

$$\delta^2\beta^2 = \beta^2 b^2 - r_s\delta \rightarrow \tag{3}$$
$$b = \delta\sqrt{1+\frac{r_s}{\delta\beta^2}}. \tag{4}$$

Two limits:

$$\frac{r_s}{\delta\beta^2} >> 1 \qquad \rightarrow b \sim \frac{\sqrt{\delta r_s}}{\beta} \tag{5}$$
$$\frac{r_s}{\delta\beta^2} << 1 \quad \rightarrow b \sim \delta\left(1+\frac{r_s}{2\delta\beta^2}\right) \tag{6}$$

Example is DM particle hitting the Sun; $r_s = 3,000\mathrm{m}$, $\beta = 10^{-3}$, $\delta = 7\times 10^8\mathrm{m}$, then $r_s/\delta\beta^2 = 4.2$, so $b = 2\delta$. At the DCA, $\beta' = \delta\beta/b = 2\beta$.

We can also start with the relativistic orbit equation,

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{m^2c^2} - \left(1-\frac{r_s}{r}\right)\left(c^2+\frac{h^2}{r^2}\right), \tag{7}$$

with $h = L/m = vb$. At the distance of closest approach (DCA) $r = \delta$, $dr/d\tau = 0$, which gives,

$$0 = \gamma^2 - \left(1-\frac{r_2}{\delta}\right)\left(1+\frac{b\beta^2}{\delta^2}\right).$$

Solving for b:

$$b = \frac{\delta}{\beta}\sqrt{\frac{\gamma^2}{1-r_s/\delta}} \tag{8}$$
$$\sim \frac{\delta}{\beta}\sqrt{\beta^2+\frac{r_s}{\delta}}, \tag{9}$$

as in the non-relativistic case.