MEMORANDUM

To: Adam Bernstein, Roy Schwitters

From: Peter Fisher

Subject: Statement of problem: looking for a signal against an uncertain background

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Detection of a signal of rate s against a background of rate b constitutes an important problem in radiation detection. In a static case, b is very well known and Poisson statistics may be used to find the measuring time t_o needed to establish $s \geq s_o$ at some level of confidence.

Seeking a radiation source in a changing background environment presents a more difficult problem. As a concrete example, consider an aircraft sampling the upper atmosphere for signs of a nuclear test. The background rate may be very poorly known and must be measured during the flight. Turning the instruments on and recording the background rate far from the test site allows the measurement of b counts with some uncertainty σ_b. After some measurement time t_b, the aircraft enters the region where radioisotopes from the test are present and the count rate increases. The aircraft then measures r counts in a time t_r. The number of signal counts is $s = r - b$ and the rate of signal counts is $\mu_s = \mu_r - \mu_b = r/t_t - b/t_b$.

We want to establish the signal rate at some level of confidence $y = \mu_s/\sigma_{\mu_s}$. We can adjust t_r and t_b or, equivalently, number of acquired signal counts s and the ratio of measurement times $\kappa = t_b/t_r$. Then,

$$\sigma^2_{\mu_s} = \frac{r}{t_r^2} + \frac{b}{\kappa^2 t_r^2} \tag{1}$$

$$= \frac{\kappa^2 \mu_s + \left(\kappa^2 + 1\right) \mu_b}{\kappa^2 t_r}. \tag{2}$$

Note that

- As $t_r \to \infty$, $\sigma^2_{\mu_s} \to 0$.
- As $\kappa \to \infty$, $\sigma^2_{\mu_s} \to \frac{\mu_s + \mu_b}{t_r} = \frac{r}{t_r^2}$.

Then,

$$y^2 = \frac{\sigma^2_{\mu_s}}{\mu_s^2} = \frac{\mu_s^2 \kappa^2 t_r}{\kappa^2 \mu_s + \left(\kappa^2 + 1\right) \mu_b}. \tag{3}$$

$$= \frac{\eta s \kappa^2}{\kappa^2 \left(\eta + 1\right) + 1} \tag{4}$$

where $\eta = \mu_s/\mu_b$ is the signal to background ratio and $s = \mu_s t_r$ is the number of signal counts measured. Solving for κ gives

$$\kappa^2 \quad = \quad \frac{y^2}{y^2\left(\eta+1\right)-\eta s} \tag{5}$$

Eq. **??** has two interesting consequences. Requiring the denominator to be positive means $\eta > y^2/s - y^2$, which implies $s > y^2$. This just says if we want a significance of $y = 3$, we need to acquire 9 signal counts; the number needed if the background is perfectly known. This makes sense as this analysis assumes the Gaussian regime and $r, b > 10$.

Second, inspection of Eq. **??** reveals for $y = 3$ that $\kappa \sim 1$ obtains over most of of the allowed (η, s) space, Fig. **??**. Except for narrow ranges of parameter space, there is no advantage to spending more time measuring background than signal.

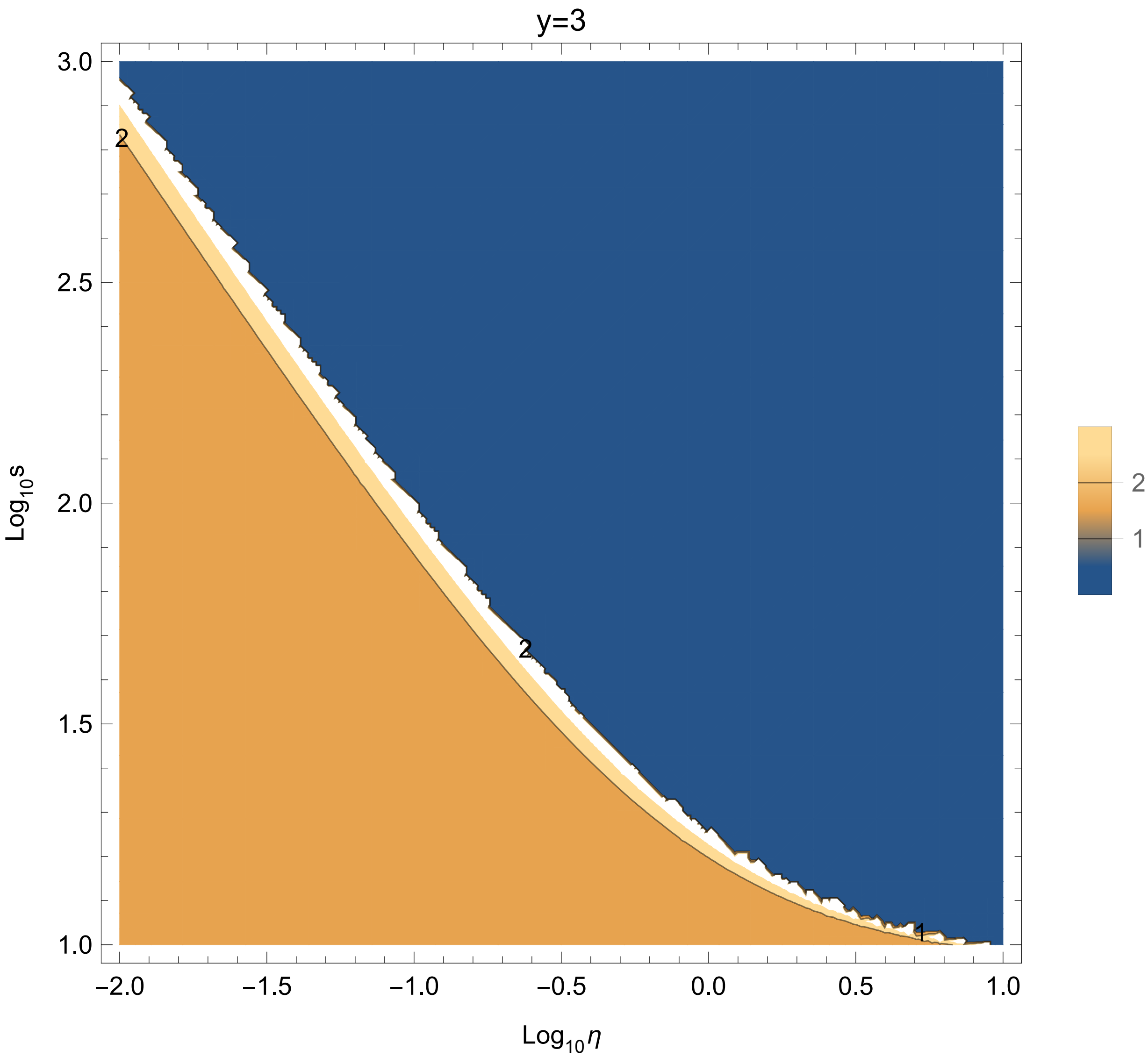

Figure 1: Factor $\kappa = \mu_s/\sigma_{\mu_s}$ plotted over (η, s) space. The blue region requires no background measurement.