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MEMORANDUM

To: Paul Dimotakis

From: Peter Fisher

Subject: Consideration of the curvature of the Earth in surveying

Date: September 10, 2018

"The Measure of All Things" by Ken Alder describes a survey of a segment of a line of longitude from Barcelona to Dunkirk. The ultimate goal was to define the meter and the survey was carried out by first measuring an interlocking system of triangles between the two cities and then precisely measuring *one* baseline in the system. The effort ultimately succeeded, but not in the way intended.

During the survey, did the curvature of the Earth play a role in the measurement of the individual triangles, given that the precision of the Borda Repeating Circle was one second of arc, 1" [1]?

Think of a surveying telescope mounted a distance h above the ground with N survey targets of height h equally spaced along a circle of constant radius around the survey telescope an define $R = R_E + h$. Will the measured the angles between each successive pair of targets add up to 2 π? The answer is, "No" if the actual angles between the targets is measured, "Yes" is the azimuthal angles are measured, Fig. 1.

When taking a site, the surveyor aims the survey telescope at the target. In the case of a flat Earth, the elevation angle, ψ in Fig. 2, will be perpendicular to a vector normal to the Earth, but for a spherical Earth, the elevation angle will be negative, as shown in Fig. 2, so a sum of the angles between pairs of survey markers will add up to less than 2 π. For example, a survey telescope at the North Pole will measure a and angle of $\pi/4$ between two survey markers on the equator separated by $\pi/2$. Adding up the pair-wise angles between four such markers on the equator will give $\pi = 4 \times \pi/4$. Adding up the azimuthal angles alone always totals 2π for survey markers arranged on a circle.

Start at the North Pole and using a measuring wheel to measure out a radius $\rho = R\theta$ along the Earth's surface. Then go around a meridian of constant ρ from the North Pole, leaving N survey markers along the way. The circle will measure,

$$C = 2\pi R \sin\theta = 2\pi l \left(1 - \frac{l^2}{4R^2}\right)$$

and the distance between each pair of successive survey stations is C/N, the angle subtended by each pair is C/lN and adding up all N gives,

$$\frac{C}{l} = 2\pi \left(1 - \frac{l^2}{4R^2}\right) < 2\pi.$$

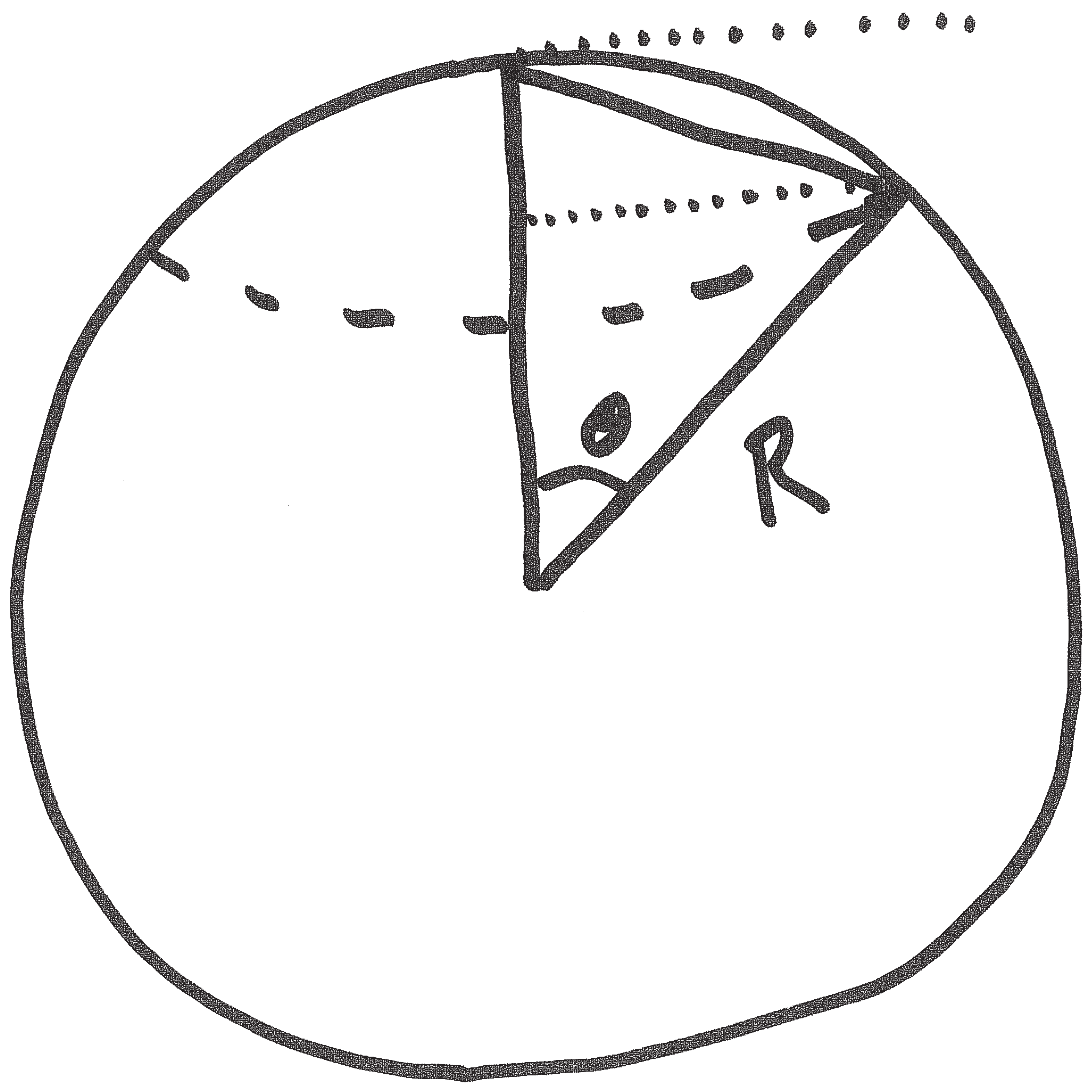

Figure 1: Survey telescope at measuring targets along a circle whose radius subtends an angle θ from the center of the Earth.

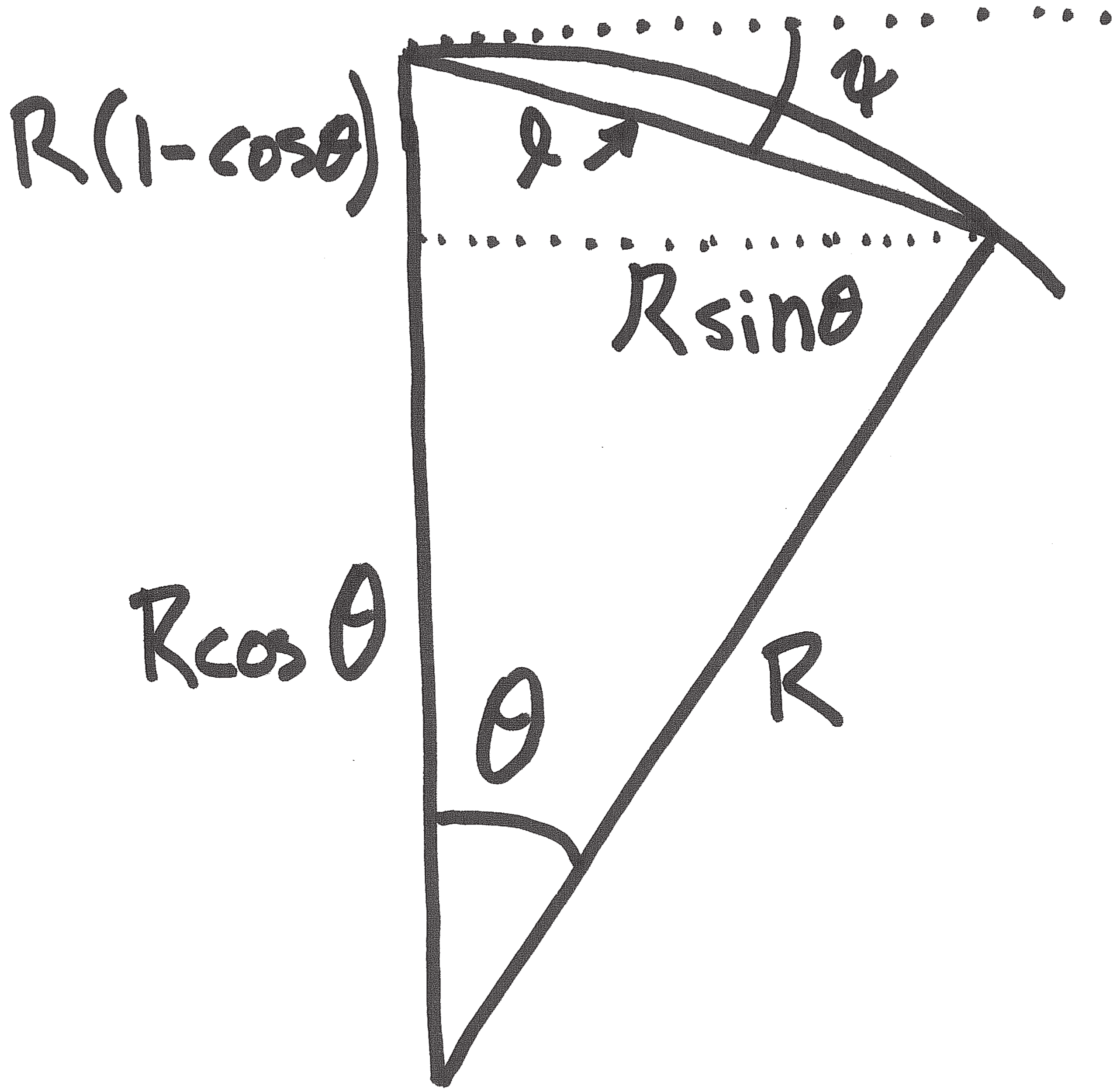

Figure 2: Detail of the cross section of the Earth.

The elevation angle works out to be $\psi \sim -\theta/2 \sim -l/2R \sim -77\mu\mathrm{rad} \sim -8$", probably much more than the precision of the elevation measurement on an eighteenth centruy survey telescope.

For the 1792-1799 measurement campaign, $l \sim 1$ km and $R = 6,438$ km, so the variation from 2π in any single triangle was $l^2/4R^2 \sim 6 \times 10^{-9}$ or 6 ppb. The *azimuthal* precision of the Borda Repeating Circle was 1″=4.8 μ rad, so the curvature of the Earth would not be an important correction for any individual triangle, especially when compared with corrections needed for the local topography. Also, the Borda circle only worked on the azimuthal angle [2] - the elevation angle measurement was much lower precision.

The 1792-1799 campaign tiled the Earth's surface with locally flat triangles to measure the distance along a line of longitude from Barcelona to Dunkirk and used measurements of latitude at both cities find the fraction of the quadrant of the Earth needed to complete the definition of the meter. All the precision came from the azimuthal measurement and the elevation measurements were corrected to $\psi = 0$. The Earth's physical shape played a dominant role in the fate of the campaign, leading to an unexpected result.

References

[1] https://physicstoday.scitation.org/doi/full/10.1063/1.1603081

[2] https://en.wikipedia.org/wiki/Repeating_circle