# MEMORANDUM

**To:** Paul Dimotakis

**From:** Peter Fisher

**Subject:** Orbit calculations between two points on Earth

**Date:** April 3, 2019

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The problem is to find the orbit equation passing through two points on the Earth's surface. We know the latitude and longitude of each point. There is a vector pointing from of the Earth to each:

$$\vec{x}_i = \left(R_E \cos\phi_i \sin\theta_i, R_E \sin\phi_2 \sin\theta_i, R_E \cos\theta_i\right).$$

The chord length from 1 to 2 is $l = |\vec{x}_1 - \vec{x}_2|$. The chord defines an isosceles triangle with $\vec{x}_1$ and $\vec{x}_2$. Bisecting this triangle gives two right triangles with hypotenuse $R_E$, base $l/2$, and angle $\sin\xi/2 = l/2R_E$. The area of the triangle is,

Define $\vec{A} = \vec{x}_1 \times \vec{x}_2$. The vector $\vec{h} = \vec{A} \times \vec{x}_1$ points in the direction of the Great circle from 1 to 2 and the vector $\vec{k} = \hat{z} \times \vec{x}_1$ points East, so $\vec{n} = \vec{x}_1 \times \vec{k}$ points North, and the heading is $\psi = \cos^{-1} \vec{n} \cdot \vec{h} / |n|\,|h|$.

The missile's flight will follow a ellipse with one focus at the center of the Earth and the apogee of the missile's flight will be the semi-major axis of the missile's elliptical orbit. The orbit equation is

$$r = \frac{p}{1 + \epsilon\cos\theta},$$

with $p = H^2/\mu$, $H = r^2/\dot{\theta}$, and $\mu = G\left(m_E + m_r\right) \sim Gm_E = 3.896\times10^{14}\ \mathrm{m}^3/\mathrm{s}$. At the apogee of the missile's flight, $\theta = 0$ and $r = R_E + a = p/1 - \epsilon$ and at Points 1 and 2, $r = R_E = p/1 + \epsilon\cos\pi - \xi/2$ where $a$ is the altitude of the missile above the Earth's surface at apogee. Solving gives,

$$\epsilon = \frac{a}{a + R\left(1 - \cos\xi\right)} \tag{1}$$

$$p = \frac{R\left(a + R\right)\left(1 - \cos\xi\right)}{a + R\left(1 - \cos\xi\right)} \tag{2}$$

so we choose $a$ and find $p$ and $\epsilon$. The launch angle is $dr/d\theta$ at Point 1,

$$\frac{dr}{d\theta} = -\frac{p\epsilon\sin\theta}{\left(1 + \epsilon\cos\theta\right)^2} = -\frac{r^2\epsilon\sin\theta}{p}$$

and at Point 1,

$$\tan\theta_l = \frac{R_E\epsilon}{p}\sin\xi.$$

| City | Distance (km) | Heading (deg.) | Launch (deg.) | $V_{BO}$ (km/s) | Flight time (s) |
|---|---|---|---|---|---|
| Washington DC | 11,082 | 17.83 E | 20.04 | 7.362 | 4,089 |
| New York City | 10,969 | 14.94 E | 20.55 | 7.346 | 4,083 |
| Boston | 10,848 | 12.40 E | 20.49 | 7.329 | 4,067 |
| Chicago | 10,444 | 24.262 E | 20.49 | 7.268 | 4,011 |
| San Francisco | 9,027 | 47.98 E | 24.62 | 7.023 | 3,852 |
| Los Angeles | 9,573 | 48.52 E | 23.57 | 7.125 | 3,917 |
| Moscow | 6,432 | 41.67 W | 30.66 | 6.39 | 3,598 |
| Berlin | 7,959 | 36.44 W | 27.31 | 6.79 | 3,738 |
| London | 8,691 | 30.06 W | 26.91 | 6.96 | 3,820 |
| Tehran | 6,407 | 68.05 W | 31.30 | 6.379 | 3,603 |

Table 1: Launch information for representative cities.

As a worked example, take Point 1 to be Pyongyang (39°15′38.74″ N, 125°51′43.33″ E) and Washington DC (38°54′25.89″ N, 77°2′12.73″ W) . From the equation for $l$ above, $l = 8,205$ km, $\xi = 1.739$, and the distance along the Earth's surface is 11,081 km. The heading is $17.83°$

Fig. 1 shows the initial velocity and apogee as functions of launch angle. Burnout velocity, $V_{bo}$ is a key constraint for missiles and shows a minimum at 22°, which gives an apogee altitude of 1,100 km. Fig. 2 shows a trajectory over the Earth's surface. Table 1 shows trajectory information for representative cities.

Another application is given by the flight of the Hwasong-15 missile on Nov. 28, 2017. The Hwasong-15 travelled 950 km toward Japan and achieved an apogee of 4,475 km [1]. In the flat Earth approximation,

$$R = \frac{2v_o^2 \sin 2\theta_l}{g} \tag{3}$$

$$y_{apogee} = \frac{v_o^2 \sin^2 \theta_l}{g} \tag{4}$$

and solving these gives $v_o = 6.636$ km/s and $\theta_l = 86.80°$. Using the orbit equations given above, $v_o = 7.196$ km/s and $\theta_l = 84.56°$. This points out the need to use the orbit equations even for ranges of a few hundred kilometers.

## References

[1] https://en.wikipedia.org/wiki/Hwasong-15#cite_note-reuters.com-9

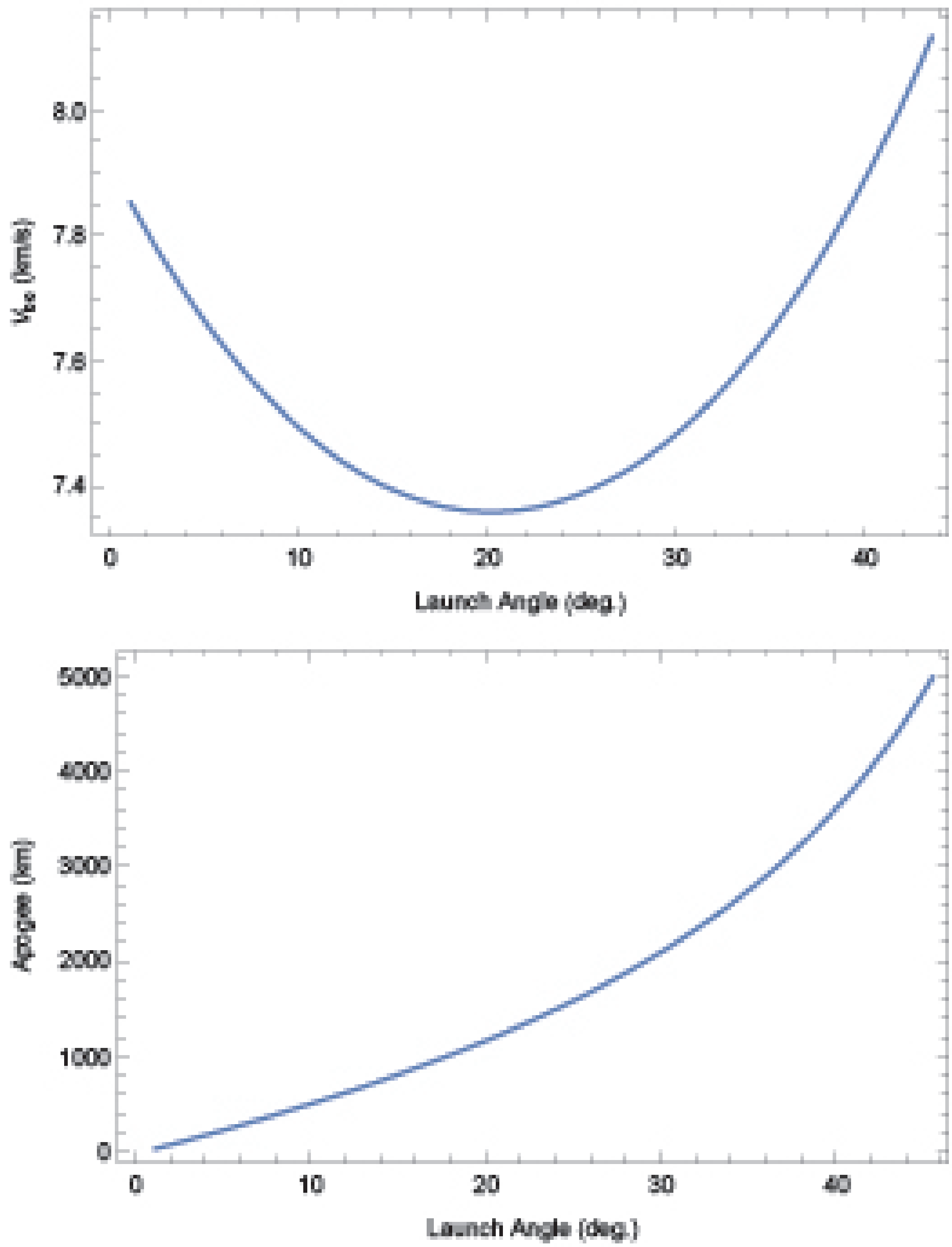


Figure 1: Upper panel shows $V_{bo}$ as a function of launch angle for a Pyongyang to Washington DC trajectory. Lower panel shows altitude at apogee.

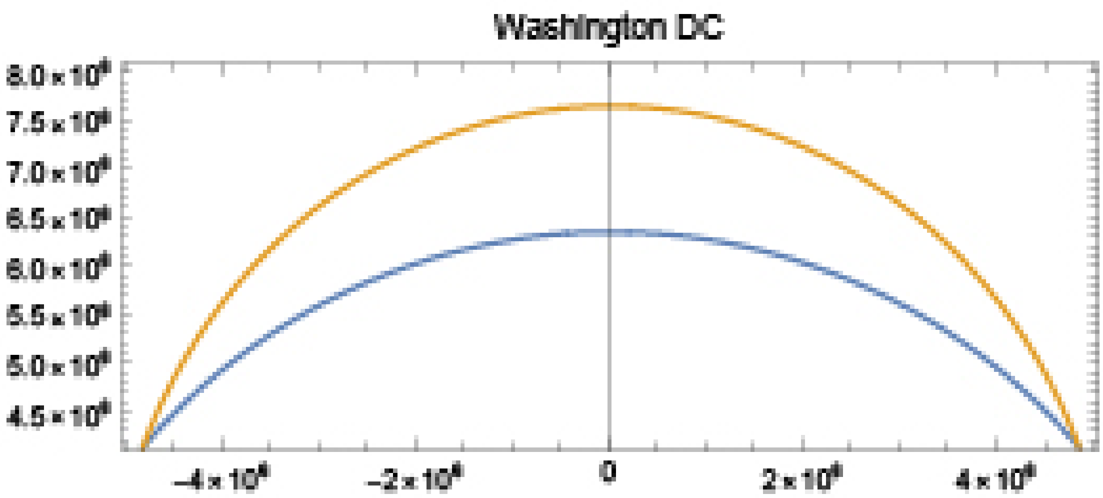


Figure 2: Minimum $V_{bo}$ trajectory over the Earth's surface from Pyongyang to Washington DC. Dimensions are in meters.