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MEMORANDUM

To: ALCON

From: Peter Fisher

Subject: Optimal Staging

Date: August 20, 2018

The memo works out the optimum masses ratios for a two stage rocket with stage masses m_1 and m_2 and payload mass m_p. Each stage carries a fraction γ of it mass as propellant, $\alpha = m_2/m_1$ and $f = m_p/m_1$. Each stage has the same fuel with specific impulse I_{sp}. The Tsoilkovsky equation gives the velocity increment for each stage,

$$\Delta v_1 = gI_{sp} \ln \frac{m_1 + m_2 + m_p}{m_1 + m_2 + m_p - \gamma m_1} \tag{1}$$

$$= gI_{sp} \ln \frac{1 + \alpha + f}{1 + \alpha + f - \gamma} \tag{2}$$

$$\Delta v_2 = gI_{sp} \ln \frac{m_2 + m_p}{m_2 + m_p - \gamma m_2} \tag{3}$$

$$= gI_{sp} \ln \frac{\alpha + f}{\alpha + f - \gamma\alpha} \tag{4}$$

$$\Delta v = \Delta v_1 + \Delta v_2. \tag{5}$$

Choosing α to maximize Δv may be a good thing to do. Then,

$$\frac{d\Delta v}{d\alpha} = \left[-\frac{\gamma \left(f + 2\alpha f^2 + f(1+f)(f-\gamma) + \alpha^2(-1+f+\gamma) \right)}{(\alpha+f)(1+\alpha+f)(-f+\alpha(-1+\gamma))(1+\alpha+f-\gamma)} \right] \tag{6}$$

$$\tag{7}$$

is zero when,

$$\alpha = \left[\frac{-f^2 - \sqrt{f - 2f\gamma - f^2\gamma + f\gamma^2 + f^2\gamma^2}}{-1 + f + g} \right]. \tag{8}$$

The argument in the square root must be positive, giving an upper limit on the payload fraction of

$$f < \frac{1-g}{g}$$

for there to be a minimum. This tells use there is, in general, a value for α for a given m_T, m_p and f that gives the maximum ΔV. Usually, it is easier to find the maximum numerically rather than use Eq. 8.

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1 The Hwasong 15

The Hwasong 15 flew a high altitude trajectory in Nov. 2017, landing 950 km to the east of its launch point achieving a velocity of 7.2 km/s [2]. Several authors created models to simulate a flight to CONUS [2, 4, 5]. How much difference do the models make? Table 1 gives the parameters for the different models.

Parameter	Unit	1800-65-30a	Hwasong-15		
			JASON	Postol[4]	Savelsberg[5]
Stage 1 mass	kg	1170	53,700	38,000	36,722
Stage 2 mass	kg	487	17,897	7,729	8,905
Stage 3 mass	kg	142.9	-	-	-
Payload mass	kg	30	500	690	1,257[1]
Stage 1 length	m	4.3	13.5	Not given	Not given
Stage 2 length	m	1.98	4.5	Not given	Not given
Stage 3 length	m	0.5	-	-	-
Rocket diameter	m	0.5	2.4	1.2	2.25
Stage 1 propellent fraction	%	89.8	88	90	90
Stage 2 propellant fraction	%	89.8	88	87	90
Stage 3 propellant fraction	%	79.0	-	-	-
Stage 1 burn time	s	21.2	150	115	130
Stage 2 burn time	s	21.2	100	185	161
Stage 3 burn time	s	137.6	-	-	-
Stage 1 fuel		HTPB	UDMH	UDMH	UDMH
Stage 2 fuel		HTPB	UDMH	UDMH	UDMH
Stage 3 fuel		UDMH	-	-	-
C_D		0.2	0.2	0.2	Not given

Table 1: Parameters used in this study.

Maximizing $\alpha = m_2/m_1$ for each model gives the results in Table 2 and Fig. 1. Optimization for ΔV through α does not help much, typically increasing the fly-out velocity by 5-10%, which can be decisive. $\Delta V = 7.3$ km/s is required to reach the most distant CONUS targets [2] and a numerical analysis of the model in [4] indicates the Hwasong-15 is capable of reaching these targets.

Model	Total mass (kg)	Payload Mass (kg)	α	ΔV (km/s)	α_{opt}	ΔV_{opt} km/s
JASON	72 000	500	0.331471	9.27283	0.0966549	9.97407
Postol	46 419	690	0.203395	9.07853	0.150234	9.48948
Savelsberg	46 884	1257	0.242498	8.16648	0.214172	8.49569

[htb]

Table 2: Comparison of different models and the optimized rocket for that model's total and payload mass.

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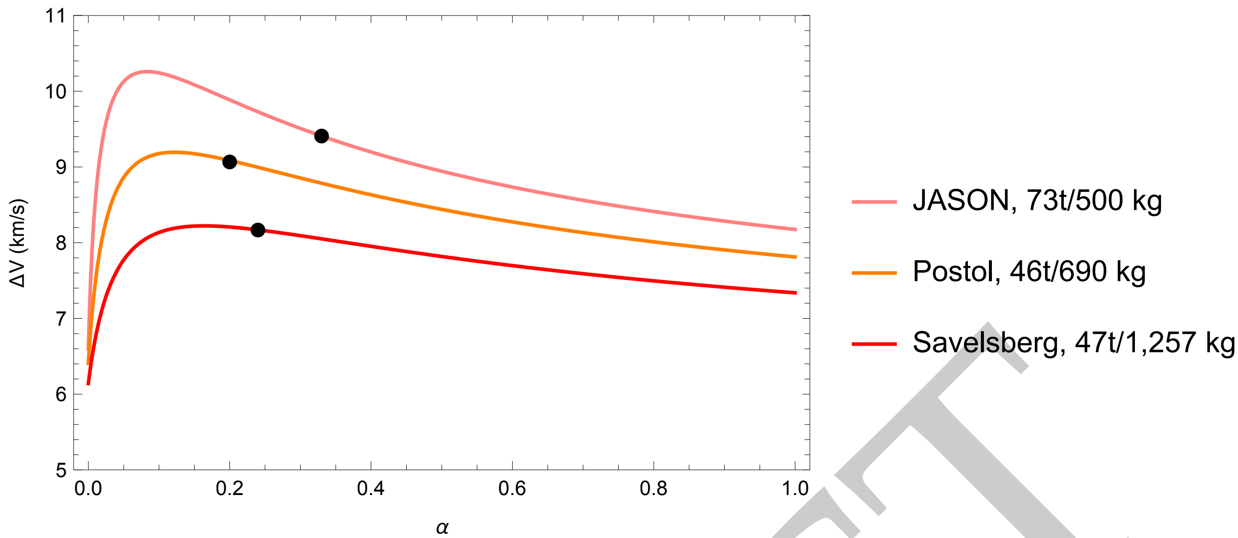

Figure 1: ΔV as a function of α for each model. The black dot indicates that model's choice for α.

2 1800-X-30 family of Interceptors

The 1800-X-30 is a family of interceptors under consideration for a specific application requiring a high ΔV. The 1800-X-30 is a three stage rocket with a total mass of 1,800 kg and a payload of 30 kg. The first two stages are HTPB solid fuel and the third is UDMH. The stages of the rocket may be completely specified by m_T, m_p, and

$$\alpha_1 = \frac{m_2 + m_3}{m_1} \tag{9}$$

$$\alpha_2 = \frac{m_3}{m_2}. \tag{10}$$

Next, $\alpha_{1,2}$ are adjusted to maximize $\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3$ where,

$$\Delta V_i = \frac{\sum_{j=i}^{3} m_j + m_P}{\sum_{j=i}^{3} m_j + m_P - g_i m_i},$$

with g_i being the propellant fraction of the i^{th} stage. Then,

$$m_1 = \frac{m_T - m_P}{1 + \alpha_1} \tag{11}$$

$$m_2 = \frac{\alpha_1}{\alpha_2 + 1} m_1 \tag{12}$$

$$m_3 = \alpha_2 m_2. \tag{13}$$

Fig. 2 shows the $\alpha_{1,2}$ parameter space and Table 3 compares the optimized values for $\alpha_{1,2}$ with those reached via other considerations. They are remarkably close, with the fly-out velocity differing by 20 m/s against 8.4 km/s.

At least in these two cases, optimization of the staging does not seem to dominate the design of a rocket one m_T and m_P have been specified. Staging optimization is a good exercise during the design process, but narrowly traded against other imperatives.

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Parameter	Optimized for high $\triangle V$	Realistic
α_1	0.404224	0.512821
α_2	0.206781	0.232033
m_1(kg)	1260.48	1170.
m_2(kg)	422.212	487.
m_3(kg)	87.3056	113.
γ_1	0.88	0.88
γ_2	0.88	0.88
γ_3	0.8	0.8
Isp1	245	245
Isp2	274	274
Isp3	333	333
$\triangle V_1$(km/s)	2301.85	2039.64
$\triangle V_2$(km/s)	3136.51	3064.87
$\triangle V_3$(km/s)	2955.97	3267.15
$\triangle V_T$(km/s)	8.39433	8.37166

[htb]

Table 3: Parameters for an optimized rocket with total mass m_T and payload mass m_P compared with the design parameters for the 1800-65-30a.

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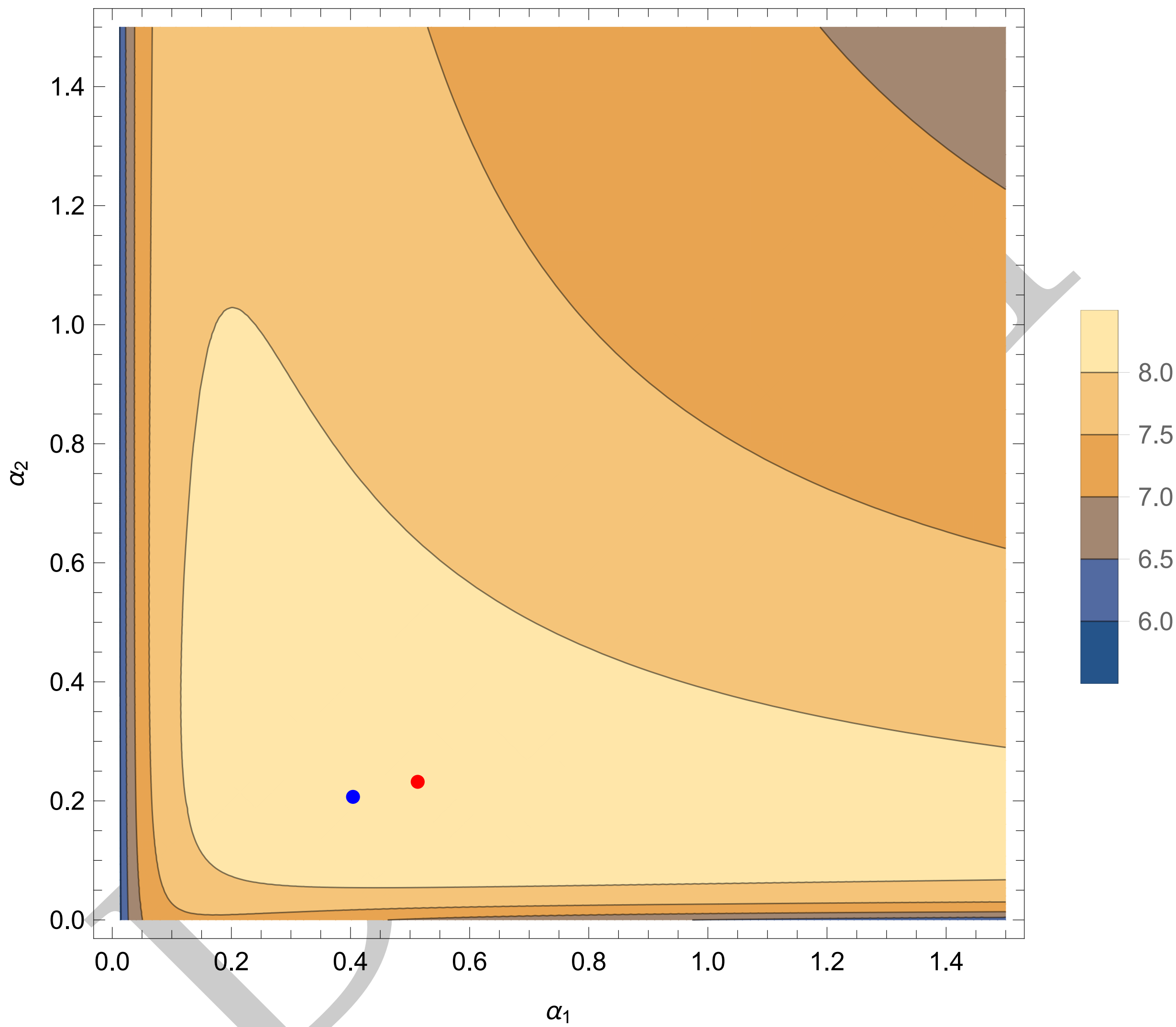

[htb]

Figure 2: ΔV contour plot in α_1, α_2 space for $m_T = 1,800$ kg, $m_P = 30$ kg. The blue dot indicates the optimum point, the red dot indicates the point evolved in the 1800-65-30a.

References

[1] http://www.braeunig.us/space/propel.htm.

[2] Fisher, P, "Orbit calculations between two points on Earth", memo from July 16, 2018.

[3] http://www.astronautix.com/n/n2o4udmh.html.

[4] Postol, T, "Pitch-over Data for Hwasong-15 Powered Flight Calculation", email to R. Garwin, July 19, 2018.

[5] Savelsberg, R.,"The DPRK's Hwasong-15: Towards a credible deterrent?", Netherlands Defence Academy, 18 April 2018. The parameters for Version 1 of their rocket is given.

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