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MEMORANDUM

To: ALCON

From: Peter Fisher

Subject: Proof that a pentagon can be inscribed in a circle

Date: November 13, 2017

Using the rules of construction, can a pentagon be inscribed in a circle?

Fig. 1 shows the layout of the problem. The segments AB, CD, and DA all have length x. Do the segments BE and CE also have length x? Alternatively, can one prove that $\theta = 2\pi/5$? Ptolemy's theorem says,

$$AC \cdot BD = BC \cdot AD + AB \cdot CD.$$

If $AC = BD = BC = y$ and $AB = CD = AD = x$,

$$y^2 = x^2 + xy.$$

Take $x = 1$ and $y = \left(1 \pm \sqrt{5}\right)/2$. Since length must be positive, choose positive root, $y = \left(1 + \sqrt{5}\right)/2 = BP$. If the segment $BE = x = 1$, then $\sin\phi = y/2 = \left(1 + \sqrt{5}\right)/4$.

The angle $OBP = \pi/2 - \theta$, the angle $EBP = \pi/2 - \phi$ and these two angles must sum to ϕ, so $\phi = \pi - \theta - \phi$, giving $\theta = \pi - 2\phi$. $\cos\theta = \cos\left(\pi - 2\phi\right) = \cos\left(2\phi - \pi\right) = -\cos 2\phi = \left(\sqrt{5} - 1\right)/4$.

Consider the vector $OE = (0, 1)$. Rotating OE by $5\theta/2$ should give a vector $OQ = (0, -1)$. Rotate first by 2θ,

$$\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix},$$

then by $\theta/2$,

$$\begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix},$$

and apply to $(0, 1)$ to get,

$$\begin{pmatrix} \cos 2\theta\cos\theta/2 - \sin 2\theta\sin\theta/2 \\ \cos 2\theta\sin\theta/2 + \sin 2\theta\cos\theta/2 \end{pmatrix}.$$

Finally, relate everything back to $\cos\theta$ using,

$$\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\sin 2\theta &= 2\cos\theta\sin\theta \\
\sin\frac{\theta}{2} &= \sqrt{\frac{1}{2} - \frac{1}{2}\cos\theta} \\
\cos\frac{\theta}{2} &= \sqrt{\frac{1}{2} + \frac{1}{2}\cos\theta} \\
\sin\theta &= \sqrt{1-\cos^2\theta}.
\end{aligned}$$

The result is that $\theta = 2\pi/5$, so every side must subtend the same angle.

The rules of construction are,

1. Specify O and measure out one unit.
2. Make a right triangle with 4 units along the base and 1 unit of elevation. The hypotenuse will measure $\sqrt{5}$ units.
3. Make a straight line, measure out 1 unit and $\sqrt{5}$ units. Quadrasect the line to get $y/2$.
4. Draw a straight line, cross with a perpendicular line and rule off $y/2$ on each side to give points B and C. The intersection will be the point P.
5. Set the compass to one unit, center on B and ruile off points E and A. Center on C and rule off D.
6. Center the compass on O, put the compass on C and draw the circle.
7. Connect $ABECD$.

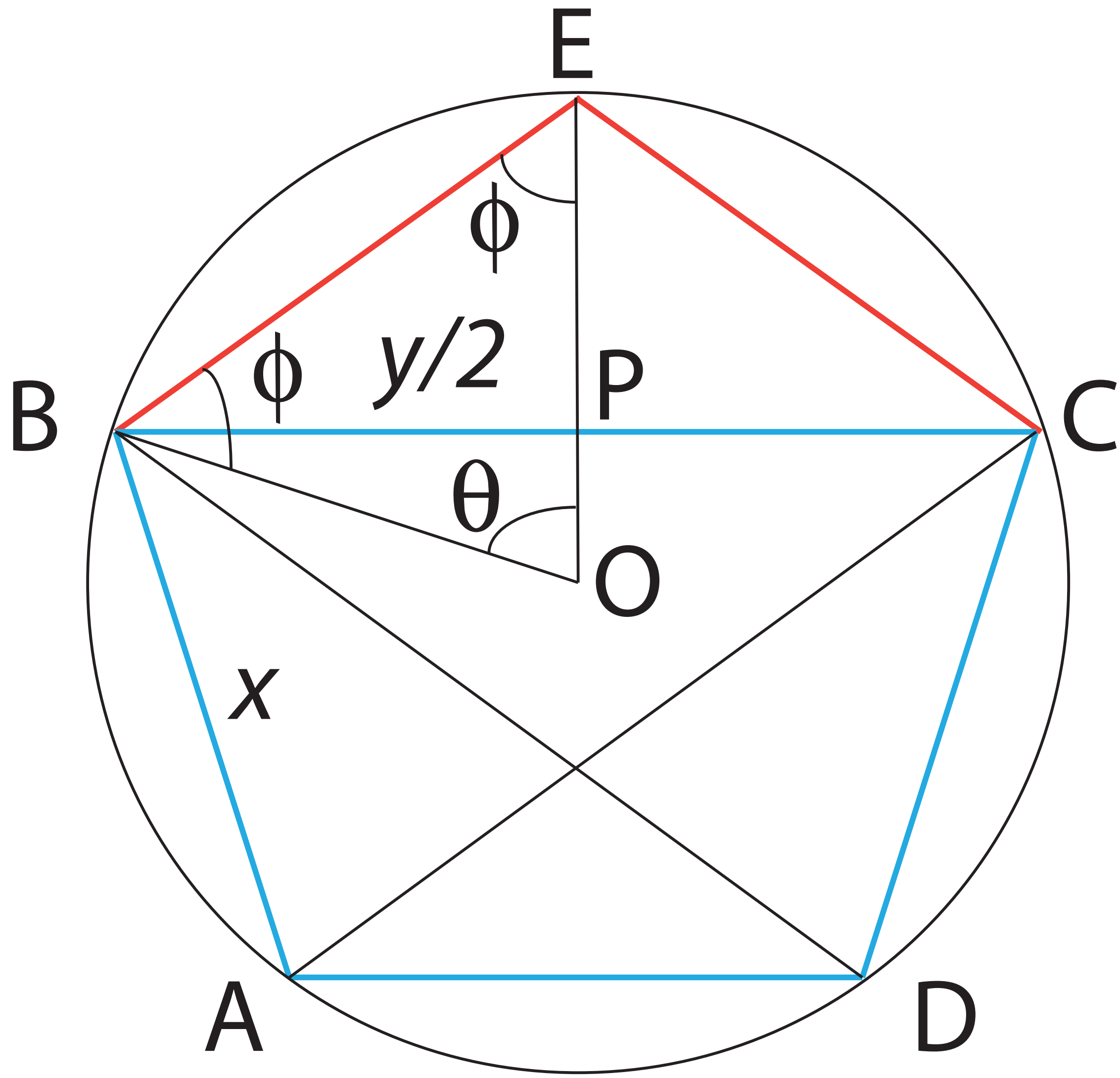

Figure 1: Layout for showing a pentagon can be inscribed in a circle.