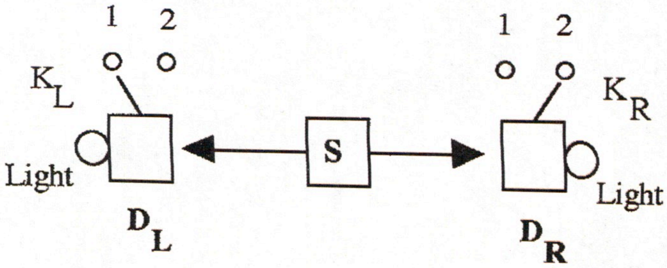
scattered photon #1, respectively); similarly for photon #2. Using this fact and the symmetry of the 2-photon final state, prove that the coincidence rate for detection of the two scattered photons is proportional to $13 - 8\cos^2\theta$. In deriving this result it is essential to recognize that the scattering process (photon #1 → photon # 1') is incoherent with respect to the scattering process (photon #2 → photon # 2'). Why is this so?

12. In this problem we consider a thought experiment proposed only very recently by L. Hardy (1993). It illustrates the peculiar and counter-intuitive nature of entangled states in a different and in some ways simpler manner than the usual Bell's inequality experiments, which employ atomic cascades and 2-photon polarization correlations. The present thought experiment makes use of a source S and two detectors D_L and D_R (L,R for left, right respectively; see the figure).



Each detector has two modes 1,2 determined by the position of a switch $K_{L,R}$. Each detector is equipped with a light that can flash either green or red. An experimental trial commences when the observer presses a button that launches a pair of correlated particles from source S; one particle goes to the left and the other to the right. After they have been emitted from the source but before they arrive at their respective detectors, the observer flips one coin to determine the position of K_L, and another coin to determine the position of K_R. The arrival of a particle at D_L is indicated by the flashing of the green or red light there; similarly for the arrival of the other particle at D_R. The outcome of a given trial is specified by giving the positions of the two switches and the color of lights which flashed; for example (1G2R) signifies that K_L was in position 1 and D_L flashed green, while K_R was in position 2 and D_R flashed red.

The observer repeats the experiment, writing down the outcome for each trial, and finds the following results after many trials:

i) When both switches are in position 1, both lights never flash red: (1R1R) never occurs.

ii) When the switches are in different positions, both lights never flash green:

(1G2G) and (2G1G) never occur.

iii) In a non-zero fraction of the trials when both switches are in position 2, the lights both flash green: (2G2G) sometimes occurs.

It is tempting to try to make the following (classical) analysis: Something in the common origin of the particles must be responsible for the observed correlations. Since the switches $K_{L,R}$ are not set until after the particles leave the source, whatever features the particles possess cannot depend on how these switches are set. Furthermore we can safely assume that D_L can only respond to the particle on the left, while D_R can only respond to the particle on the right. Then, since any trial could end up as a 12 or a 21 trial, whenever one of the particles is of a variety to allow a type 2 detector to flash green, the other particle must be of a variety to make a type 1 detector flash red. (This follows from ii above). Then in any of the occasional 22 trials where both detectors flashed green, both particles must have been of the variety to make a type 1 detector flash red. In other words, had both switches been set to position 1 in these trials, the outcome 1R1R would have been observed. However 1R1R is never observed! Thus the foregoing classical argument leads to a contradiction.

Nevertheless it <u>is</u> possible in principle to set up such an experiment and to get the results given, but we must use quantum mechanics to describe the system of particles. Suppose that when a switch K is set to position 1 the outcome "green" corresponds to absorption of a particle of spin 1/2 with spin up along the z axis, whereas the outcome red corresponds to spin down:

$$|1G\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |1R\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{12.1}$$

Since we never obtain the outcome (1R1R) we can assume that the quantum state of the two particles launched from the source is of the form:

$$|\psi\rangle = \alpha\,|1R\;1G\rangle + \beta\,|1G\;1R\rangle + \gamma\,|1G\;1G\rangle \tag{12.2}$$

where $|1R\;1G\rangle$ refers to left particle with spin down, right particle with spin up, and so forth, and α,β,γ are constants. We may assume that $|\psi\rangle$ is normalized to unity, so that $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$.

a) (10) Show that, since outcomes (1G2G) and (2G1G) never occur and the states $|1G2G\rangle$ and

|2G1G> are thus orthogonal to |ψ>, it follows that:

$$\alpha \langle 2G \mid 1R\rangle + \gamma \langle 2G \mid 1G\rangle = 0$$
$$\beta \langle 2G \mid 1R\rangle + \gamma \langle 2G \mid 1G\rangle = 0 \qquad (12.3)$$

b) (20) It must be possible to express |2G> as a linear combination of the states |1G>, |1R>; and |2R> must be an orthogonal linear combination:

$$|2G\rangle = q^{1/2}|1G\rangle + \sqrt{1-q}\,|1R\rangle$$
$$|2R\rangle = -\sqrt{1-q}\,|1G\rangle + q^{1/2}\,|1R\rangle \qquad (12.4)$$

where $0<q<1$. Show that, since outcome (2G2G) sometimes occurs, and therefore $|\langle 2G2G|\psi\rangle|^2 = p \neq 0$, it follows that:

$$p = \frac{q^2(1-q)^2}{1-q^2} \qquad (12.5)$$

c) (30) Show that when p is maximized in (12.5) the probabilities of the various outcomes are given by the following table, where $z = \frac{1}{2}(\sqrt{5}-1)$:

1G1G	z^3
1G1R	z^2
1R1G	z^2
1R1R	0
1G2G	0
1G2R	z
1R2G	z^3
1R2R	z^4
2G1G	0
2G1R	z^3
2R1G	z
2R1R	z^4
2G2G	$z^5 = p$
2G2R	z^4
2R2G	z^4
2R2R	z

$z^2 = \frac{1}{4}(5 - 2\sqrt{5} + 1) = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}$

$z^3 = \frac{3-\sqrt{5}}{2} \cdot \frac{-1+\sqrt{5}}{2} = \frac{-3+3\sqrt{5}+\sqrt{5}-5}{4} = \frac{-8+4\sqrt{5}}{4}$

$= -2+\sqrt{5}$

$$\vec{\sigma}\cdot\mathbf{A}\,\vec{\sigma}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{B} + i\,\vec{\sigma}\cdot\mathbf{A}\times\mathbf{B} \qquad (8.2)$$

Identities (8.1) and (8.2) are very important and will be used repeatedly in 221A,B.

9.(20) Later in the course we shall discuss the properties of the "angular momentum" operators J_x, J_y, J_z. In units where $\hbar=1$, these satisfy the commutation relations:

$$[J_x,J_y] = iJ_z, \quad [J_y,J_z] = iJ_x, \quad [J_z,J_y] = iJ_x$$

Let α be a real parameter. Show that

$$e^{-i\alpha J_z} J_y e^{i\alpha J_z} = a\,J_y + bJ_x$$

where a, b are two real quantities. Find the latter in terms of α.

10.(15) Let A(x) be an operator that depends on a continuous parameter x and let dA/dx be the derivative with respect to x. Derive the following operator identity:

$$e^{-iA}\frac{d}{dx}(e^{iA}) = i\sum_{n=0}^{\infty}\frac{(-i)^n}{(n+1)!}A^n\left\{\frac{dA}{dx}\right\}$$

where $A^0\{B\} = B$, $A^1\{B\} = [A,B]$, $A^2\{B\} = [A,[A,B]]$, etc.

11. An electron and a positron can form a bound system called positronium. In the ground $^1S_0$ state, a positronium atom decays by annihilation of e^+ and e^- into two photons with equal and opposite linear momentum in the positronium rest frame. Also, in this ground state, the positronium atom has zero total angular momentum; hence the two photons carry off zero total angular momentum. Furthermore, as will be shown in 221B, the parity of the positronium ground state is odd, which means that under inversion of spatial coordinates the state vector changes sign. Also, parity is conserved in the annihilation process, so the parity of the two-photon final state is odd.

a) (10) Consider a photon emitted along +z, and another emitted along -z. Show that the two-photon final state must be of the form: