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Quantum Radar

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We propose a quantum metrology protocol for the localization of a noncooperative pointlike target in three-dimensional space, by illuminating it with electromagnetic waves. It employs all the spatial degrees of freedom of N entangled photons to achieve an uncertainty in localization that is $\sqrt{N}$ times smaller for each spatial direction than what could be achieved by N-independent photons or by classical light of the same average intensity.

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Quantum metrology [1–5] is a set of procedures that increase the precision in the estimations of parameters by employing quantum effects such as entanglement or squeezing. By entangling N different probes, typical protocols achieve a $\sqrt{N}$ decrease in the statistical noise over what would be achievable without entanglement. Here we will present a quantum metrology protocol for a radar. Radar stands for radio detection and ranging, so the bare minimum for a protocol to qualify as such is that it is able to detect a target and return its position relative to the receiver. However, previous quantum radar protocols [6] based on quantum illumination [7] fail this requirement, as they can only discriminate whether the target is present or not, and they give a quantum advantage only in the presence of a rather specific thermal noise model. Other protocols [8,9] still are unable to provide *both* detection and position of the target with enhanced precision. In this Letter, we will present a quantum metrology protocol for a radar. Instead, our protocol returns both and does not require the target to cooperate. It achieves an $N^{3/2}$ decrease in the uncertainty volume of the target position over what could be achieved with N-independent photons of the same spatial bandwidth, namely, a $\sqrt{N}$ decrease in uncertainty along each of the three spatial dimensions. The main drawbacks of our protocol are the difficulty in creating the required entangled state of the electromagnetic field and its sensitivity to noise.

The main idea of our protocol is to combine a three-dimensional generalization of the one-dimensional quantum localization protocol of [10,11] with a free-space propagation analysis of the signal from target to receiver. The use of all the spatial degrees of freedom of the entangled photons allows three-dimensional localization. Our protocol can be used as an aid to conventional radar systems rather than a replacement.

The protocol allows a receiver to find her position relative to an uncooperating target object that is illuminated with a suitable entangled state of light composed of N entangled photons, see Fig. 1. To this aim, the receiver measures their arrival position and arrival time on a transverse plane at her location. Consider $N=2$ first. The joint probability of photodetection, namely, of finding the two photons at times t_1 and t_2 and at positions $\vec{r}_1$ and $\vec{r}_2$ (two-dimensional transverse vectors) is

$$p(t_1,\vec{r}_1;t_2,\vec{r}_2)\propto|\langle 0|E^+(t_1,\vec{r}_1)E^+(t_2,\vec{r}_2)|\psi_2\rangle|^2, \quad (1)$$

where $|0\rangle$ is the vacuum state, the proportionality constant depends on the detector's specs [12], $|\psi_2\rangle$ is the state of the two photons (we work in the Heisenberg picture, where the operators evolve from an initial time t_0), and $E^+(t,\vec{r})\equiv\int d^3k_3 g(\vec{k}_3,t,\vec{r})a(\vec{k}_3)e^{-i\omega(t-t_0)}$ (e.g., [13]), where g is the transfer function (defined below) between the object plane (at the target's position) and the image plane (at the position of the receiver). $a(\vec{k}_3)$ is the electromagnetic field annihilation operator for the mode with wave vector $\vec{k}_3=(k_x,k_y,k_z)$. As customary, we will employ the far field approximation, valid when the object-receiver distance is sufficiently large: the longitudinal component of the wave vector of the received light is much larger than the transverse components: $k_x^2+k_y^2\ll|\vec{k}_3|^2$, with $|\vec{k}_3|=(k_x^2+k_y^2+k_z^2)^{1/2}=\omega/c$ (ω is the light frequency). So we can approximate the $\vec{k}_3$ integral as $\int d^3k_3=\int(d\omega/c^2)d^2k/\sqrt{1/c^2-(k_x^2+k_y^2)/\omega^2}\simeq(1/c)\int d\omega d^2k$, with $\vec{k}=(k_x,k_y)$ as the two-dimensional transverse wave vector. Then, E^+ given above can be replaced by

$$E^+(t,\vec{r})\simeq\int d\omega d^2k g(\vec{k},\vec{r})a(\omega,\vec{k})e^{-i\omega(t-t_0)}, \quad (2)$$

where the longitudinal component contributes only with a phase factor that measures the longitudinal distance $z=c(t-t_0)$ that the light travels from the source to the target, and back to the detector, and where the free-space (transverse) transfer function is

$$g(\vec{k},\vec{r}) \equiv \int d^2 r_o A(\vec{r}_o) e^{i\vec{k}\cdot(\vec{r}_0-\vec{r})}, \qquad (3)$$

where A is the object transfer function and the integral is over the (transverse) object plane, namely, $\vec{r}_o$ and $\vec{r}$ are two-dimensional transverse vectors. We will consider a pointlike reflective object that reflects only the photons that impinge on its position $\vec{r}_p$. The other photons are lost. This situation is described by a transfer function that has value a in the vicinity of $\vec{r}_p$ in the object plane and value zero elsewhere in the object plane, namely, $A(\vec{r}_o) \propto a\delta(\vec{r}_o - \vec{r}_p)$. Slightly more general situations can be considered, but it is not possible to perform more complex imaging with entangled light since the transfer function g of any imaging apparatus is more complex than (3) and the photon correlations in (4) (below) will prevent the formation of a discernible image. For radar applications, we are only interested in free-space propagation, described by (3) for detection and ranging.

Interestingly, the pointlike approximation $A \propto \delta$ can be dropped without losing the quantum enhancement in the regime where the light spatial bandwidth σ_ψ is much larger than the object dimension σ_A. In this case, the quantum enhancement is present, but only up to a number of photons $N_{\max} \sim (\sigma_\psi/\sigma_A)^2$. When a larger number is employed, the quantum enhancement is lost: the classical unentangled strategy yields the same precision (see the Supplemental Material [14] for details). In the rest of the Letter, we retain the pointlike approximation $\sigma_A \simeq 0$.

The necessary entangled two-photon state, produced at the initial time t_0, in the far field approximation, is

$$|\psi_2\rangle \equiv \int d\omega d^2 k \psi(\omega,\vec{k})[a^\dagger(\omega,\vec{k})]^2|0\rangle, \qquad (4)$$

where $a^\dagger(\omega,\vec{k})$ creates a photon with frequency ω and transverse wave vector $\vec{k}$, ψ is the biphoton's spatiotemporal wave function, and we omit the normalization since it is a non-normalizable state as all Einstein-Podolsky-Rosen states [19]. It is a maximally entangled state in three different degrees of freedom: k_x, k_y, and ω (we will drop this assumption later). The positive correlation in $\vec{k}$, ω in (4) implies anticorrelation in the transverse position and time of arrival. So, we can describe our radar protocol in the "position representation" (see Supplemental Material [14]) by considering two photons that originate at opposite sides of the target, hit the target at ρ_p, and end up at opposite detectors at opposite times. We must suppose that at the receiver's location there is a negligible probability of seeing the photons that are not scattered by the object, namely, (4) is an approximation of the electromagnetic field valid only in the object's vicinity. This is implicit in the far field approximation since the longitudinal component of $\vec{k}_3$ is directed away from the detector.

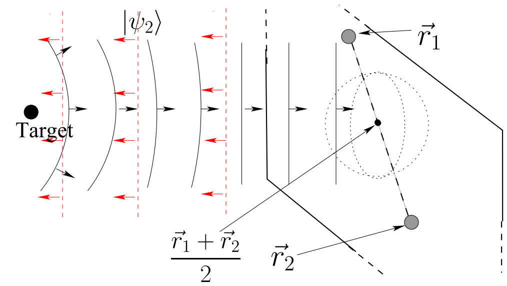

FIG. 1. Quantum radar setup (two-photon case). A pointlike target reflects the momentum-entangled state $|\psi_2\rangle$ of two photons (impinging dashed lines). In the far field, the photons arrive at a screen. The average time of arrival (not pictured) provides the longitudinal distance, whereas the average of the two photons' transverse arrival positions $\vec{r}_1$, $\vec{r}_2$ provides the object's transverse location (dashed line). The uncertainty sphere obtained (dotted line) is reduced by a factor $N^{3/2}$ over what would be obtained with N-independent photons with the same spatiotemporal bandwidth (the case $N = 2$ is depicted here).

Replacing these quantities in Eq. (1), we find

$$p(t_1,\vec{r}_1;t_2,\vec{r}_2) \propto |\tilde{\psi}(t_1+t_2-2t_0,\vec{r}_1+\vec{r}_2-2\vec{r}_p)|^2, \qquad (5)$$

where $\tilde{\psi}(t,\vec{r}) = \int d\omega d^2 k \psi(\omega,\vec{k}) e^{i(\omega t+\vec{k}\cdot\vec{r})}$ is the Fourier transform of $\psi(\omega,\vec{k})$ [20]. This implies that the average time of the arrival is equal to the transit time of the signal from its production at t_0 to its detection at t,What is the time $t=0$ physically? What is the time at which I start the clocks at the detection? We cannot use the time at which the signal interacts with the object as $t=0$, because that time in unknown! No $t=0$, which I called t_0 is the initial state when the beam is created because we're working in the Heisenberg picture. and that the average arrival transverse position is equal to the object's transverse position. The statistical noise of these two quantities is given by *half* the standard deviation of $|\tilde{\psi}|^2$ in time and in position. Indeed, the left-hand side of (5) can also be written as $|\tilde{\psi}\{2[(t_1+t_2)/2-t_0], 2[(\vec{r}_1+\vec{r}_2)/2-\vec{r}_0]\}|^2$. Hence, the standard deviation of the average time of arrival gains a factor of $1/2$ and similarly for each of the two components of the average position.

We must compare this result to what one can obtain using two unentangled photons with the same spectral characteristics. Consider a single photon in the state

$$|\psi_1\rangle = \int d\omega d^2 k \psi(\omega,\vec{k}) a^\dagger(\omega,\vec{k})|0\rangle, \qquad (6)$$

with same spectrum $\psi(\omega,\vec{k})$ as in (4). The probability of detecting it at time t at transverse position $\vec{r}$ is

$$p(t,\vec{r}) \propto |\langle 0|E^+(t,\vec{r})|\psi_1\rangle|^2 \propto |\tilde{\psi}(t,\vec{r})|^2. \qquad (7)$$

Clearly, a fair comparison must be between the two-photon entangled strategy and an unentangled strategy that uses *two* unentangled photons $|\psi_1\rangle \otimes |\psi_1\rangle$. If each of the unentangled photons provide an error equal to the standard deviation of $|\tilde{\psi}|^2$, the standard deviation of the average time of arrival gains a factor of $1/\sqrt{2}$ (as the variance of the sum is the sum of variances) and similarly for each of the two components of the average position. Thus, using the entangled state $|\psi_2\rangle$, the net gain is the square root $\sqrt{2}$ of the number of photons in the resolution along each of the three spatial directions with respect to a strategy that employs two unentangled photons $|\psi_1\rangle$.

It is easy now to extend the above discussion to an arbitrary number N of photons: the joint probability of detecting them at time t_j at transverse position $\vec{r}_j$ is

$$p(\{t_j,\vec{r}_j\}_{j=1,\dots,N}) \propto |\langle 0|\prod_j E^+(t_j,\vec{r}_j)|\psi_N\rangle|^2 \propto |\tilde{\psi}\left(\sum_j t_j - Nt_0, \sum_j \vec{r}_j - N\vec{r}_p\right)|^2, \quad (8)$$

if one uses a far field N-photon entangled state

$$|\psi_N\rangle \equiv \int d\omega d^2k \psi(\omega,\vec{k})[a^\dagger(\omega,\vec{k})]^N|0\rangle, \quad (9)$$

which describes a state where the photons are highly correlated, analogously to the two-photon case. Clearly, (8) gives a distribution that has a standard deviation for each position component and for the time of arrival that is $\sqrt{N}$ times smaller than the standard deviation obtained by averaging N unentangled photons in the state $|\psi_1\rangle$, with arrival probability (7).

Intuitively, one expects that classical light with an average photon number N will give a precision comparable to the one of N-independent single photons with the same spatial and temporal bandwidth. This is indeed true, as demonstrated in the Supplemental Material [14], showing that the $\sqrt{N}$ enhancement of the N-photon entangled state demonstrated above is the same that is obtained also with respect to a classical state with the same intensity.

We now discuss the feasibility of the experiment. For the state $|\psi_N\rangle$, the arrival time t_j and position $\vec{r}_j$ of each photon is completely random. In fact, consider the case $N = 2$: $|\psi_2\rangle$ can be written also as

$$|\psi_2\rangle = \int dt_1 d^2r_1 dt_2 d^2r_2 \tilde{\psi}(t_1+t_2, \vec{r}_1+\vec{r}_2) \times a^\dagger(t_1,\vec{r}_1)a^\dagger(t_2,\vec{r}_2)|0\rangle, \quad (10)$$

where we introduced into (4) the operator $a^\dagger(t,\vec{r}) \propto \int d\omega d^2k a^\dagger(\omega,\vec{k})e^{i(\omega t+\vec{k}\cdot\vec{r})}$ that creates a photon at time t and transverse position $\vec{r}$. Each of the two photons in (10) taken by themselves can arrive at any time and at any position, since the time and position difference have uniform probability amplitude. It is only the time and position sums (or averages) that are peaked. Indeed, the probability (5) depends only on the sums $t_1 + t_2$ and $\vec{r}_1 + \vec{r}_2$, so that the differences $t_1 - t_2$ and $\vec{r}_1 - \vec{r}_2$ must be uniformly distributed.

So, there are two main practical issues with this protocol. On one hand, it is very demanding to produce the maximally entangled states (4) and (9). On the other hand, the complete randomness in arrival times and positions requires an infinite measurement time and transverse screen. Both of these problems can be overcome by reducing the amount of entanglement among photons. This, of course, will reduce the resolution gain, but it will still allow for a better-than-classical enhancement. Again, for the sake of illustration, we will consider the case $N = 2$ first and then extend to arbitrary N.

Consider the partially entangled two-photon state

$$|\phi_2\rangle \equiv \int d\omega d^2k d\omega_d d^2k_d \psi(\omega,\vec{k})\gamma(\omega_d)\xi(\vec{k}_d) \times a^\dagger(\omega,\vec{k})a^\dagger(\omega+\omega_d,\vec{k}+\vec{k}_d)|0\rangle, \quad (11)$$

where ω_d and $\vec{k}_d$ are the frequency difference and transverse wave vector divergence between the two photons, governed by the probability amplitudes γ and ξ, respectively. The state $|\phi_2\rangle$ can and has been produced in the lab (see the Supplemental Material [14] for a review and the bibliography). It is normalizable and tends to $|\psi_2\rangle$ in the limit when γ and ξ tend to delta functions $\gamma \to \delta(\vec{k}_p)$ and $\xi \to \delta(\omega_d)$. Replacing $|\psi_2\rangle$ with $|\phi_2\rangle$ in (1), we find

$$p(t_1,\vec{r}_1;t_2,\vec{r}_2) \propto |\tilde{\psi}(t_1+t_2-2t_0, \vec{r}_1+\vec{r}_2-2\vec{r}_p)|^2 \times |\tilde{\gamma}(t_2-t_0)\tilde{\xi}(\vec{r}_2-\vec{r}_p)+\tilde{\gamma}(t_1-t_0)\tilde{\xi}(\vec{r}_1-\vec{r}_p)|^2, \quad (12)$$

where $\tilde{\gamma}$ and $\tilde{\xi}$ are the Fourier transforms of γ and ξ. In the limit in which γ and ξ are deltas, then $\tilde{\gamma}$ and $\tilde{\xi}$ are uniform, so the second line of (12) is a constant and we reobtain the maximally entangled result of (5). In the intermediate case (12) each of the photon's time of arrival t_j and transverse position $\vec{r}_j$ is limited (in contrast to the maximally entangled case), but the spread in their averages is dominated by the product between $|\tilde{\psi}|^2$ and $|\tilde{\gamma}\tilde{\xi}|^2$. For such nonmaximal entangled states (11), the standard deviation of the average time of arrival gains a factor of λ with $1/2 \le \lambda \le 1$ and similarly for each of the two components of the average position. When the bandwidth of ξ and γ is larger than that of ψ, $\lambda \le 1/\sqrt{2}$, it will always achieve a better-than-classical enhancement both in time and transverse positions.

The nonmaximally entangled N-photon version is straightforward: the state and associated probability are

$$|\phi_N\rangle \equiv \int d\omega d^2k \prod_j d\omega_j d^2k_j \psi(\omega,\vec{k})\gamma(\omega_1)\cdots\gamma(\omega_N)\xi(\vec{k}_1)\cdots\xi(\vec{k}_N)a^\dagger(\omega,\vec{k})a^\dagger(\omega+\omega_1,\vec{k}+\vec{k}_1)\cdots a^\dagger(\omega+\omega_N,\vec{k}+\vec{k}_N)|0\rangle$$

$$p(\{t_j,\vec{r}_j\}_{j=1,\ldots,N}) \propto |\langle 0|\prod_j E^+(t_j,\vec{r}_j)|\phi_N\rangle|^2 \propto |\tilde{\psi}\left(\sum_j t_j - Nt_0, \sum_j \vec{r}_j - N\vec{r}_p\right)\sum_j\prod_{n\neq j}\tilde{\gamma}(t_n-t_0)\tilde{\xi}(\vec{r}_n-\vec{r}_p)|^2, \tag{13}$$

for which considerations analogous to the case $N=2$ seen above apply. The standard deviation of the average time of arrival gains a factor of λ with $1/N \leq \lambda \leq 1$ and similarly for each of the two components of the average position. If the bandwidth of ξ and γ is larger than that of ψ, $\lambda \leq 1/\sqrt{N}$, it achieves a quantum enhancement in time and transverse positions.

The ideal state $|\psi_N\rangle$ and $|\phi_N\rangle$ for arbitrary N is actually a state that is positively correlated both in frequency *and* transverse momentum. For $N=2$, the state $|\psi_2\rangle$ has been experimentally realized under a tightly focused pulsed pump based on type II noncritical phase matching [21] (reviewed in the Supplemental Material [14]). Pulsed pumping can provide the bandwidth for the frequency correlation, and a tightly focused process can modulate the transverse momentum correlation.

We now consider the efficiency of the protocol. As customary, we compared the quantum and classical protocols for the same number of resources employed in the preparation. It is necessary to show a fair comparison for the whole protocol, namely, also after detection. We cannot directly compare the probabilities in (5) and (7), as (5) refers to the un-normalizable state (4). One needs to regularize such state, e.g., by using the normalized state $|\phi_2\rangle$ of (11) that gives probability (12). To compare these two cases, consider a specific example in which we use Gaussian weights for the regularizations γ and ξ with variance σ^2. In this case, the probability (12) is

$$p(t_1,\vec{r}_1;t_2,\vec{r}_2) \propto \sigma|\tilde{\psi}(t_1+t_1-2t_0,\vec{r}_1+\vec{r}_2-2\vec{r}_0)|^2, \tag{14}$$

whereas the probability of two single photons from (7) gives $p(t_1,\vec{r}_1;t_2,\vec{r}_2) \propto |\tilde{\psi}(t_1,\vec{r}_1)\tilde{\psi}(t_2,\vec{r}_2)|^2$. So, a comparison between these two shows that the proportionality factor in the probabilities is equal only when $|\tilde{\psi}|^2$ is of the order of σ. In this case, however, the quantum strategy is equivalent to the classical strategy as discussed above. This implies that, when using the state (11), a quantum advantage is achievable only at the cost of a reduced efficiency. Whether different entangled states (such as the nested ones proposed below) can give a better resolution is currently unknown. More details are in the Supplemental Material [14].

In this Letter, we are mainly concerned with proposing the protocol in ideal conditions. However, it is important to show that it can still give a quantum advantage when noise is considered, for example, if some photons are lost at the target (see Supplemental Material [14]). Our maximally entangled protocol is extremely sensitive to noise, as typically happens in quantum metrology: as is typical in quantum metrology [22,23], the loss of a single photon will render all the other $N-1$ ones completely useless for the estimation, since their times and positions of arrival are completely random. Many different strategies that reduce this effect at the cost of a decrease in resolution have been proposed. For example, the partially entangled state $|\phi_N\rangle$ is more robust to the loss of photons: those that do arrive still contain some information on the object position. Moreover, the strategies proposed in [10] can be adapted here: divide the N photons into subsets of M entangled photons and then entangle these subsets (a nested strategy). If one photon is lost, only the photons of its subset become useless, while those of the other subsets can still attain a quantum enhancement. Other strategies involve the use of quantum error correcting codes [24] or the use of external ancillas [25]. Even without using these techniques, a subshot noise sensitivity may still be achieved for a range of parameters: the analysis of [8] (Chap. 5.2), can be applied to our protocol, as the noise model is equivalent (the protocol is not).

In conclusion, we proposed a quantum estimation for the location of a target in three dimensions with a precision increase equal to the square root of the number of photons employed, when compared to the best unentangled strategy using photons or classic light with equal energy and spectral characteristics. Here we considered entanglement, but squeezing would work similarly [26]. As a future application, one might consider the extension of the protocol to the localization in four-dimensional spacetime to determine the spatial location and the time of an event. Unfortunately such extension is nontrivial because in electromagnetic waves the spatial and temporal degrees of freedom are connected (they are constrained by being a solution to a wave equation). So one would need a further, independent, degree of freedom to use as a clock, in addition to the photon's spatial degrees of freedom that we used here.

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