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On the status of plane and solid angles in the International System of Units (SI)

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Abstract
The article analyzes the arguments that have become the basis for the 1980 CIPM recommendations declaring plane and solid angles as dimensionless derived quantities. This decision was the result of an incorrect interpretation of mathematical relationships that connect the ratio of two lengths with the plane angle, and the ratio of area to square of length with the solid angle. The analysis of these relationships, presented in the article, showed that they determine neither the dimensions of the angles nor their units, but only the numerical values of the angles expressed in radians and steradians. It is shown that the series expansions of trigonometric functions sometimes used to prove the dimensionless character of the plane angle is also incorrect because in this case the trigonometric functions of two different types, independent of each other, are offen confused. It is established that the plane angle is an independent quantity and therefore should be assigned to the base quantities and its unit, the radian, should be added to the base SI units. It is shown that the solid angle is the derived quantity of a plane angle. Its unit, the steradian, is a coherent derived unit equal to the square radian.

Keywords: plane angle, solid angle, radian, steradian

Introduction

In November 2018, a radical reform of the International System of Units was carried out at the 26th General Conference on Weights and Measures (CGPM). Four base SI units of seven, the kilogram, the ampere, the kelvin, and the mole were redefined in a new way, based on fixing the exact values of four fundamental physical constants: Planck constant h, elementary electric charge e, Boltzmann constant k and Avogadro constant N_{A}.

At the same time, in the new edition of the SI Brochure, some changes appeared in the definitions related to the plane and solid angles, which, from 1980 to the present have been considered as dimensionless derived quantities. But it was not always so. When the SI was adopted in 1960, plane and solid angles were considered as dimensional quantities, and their units radian and steradian were included into a separate class of supplementary units. They were independent of other units and were not considered in the SI as base or derived ones.

In the framework of the SI it is assumed that the base quantities have independent dimensions, that is, none of the base quantities can be obtained from the others. Derived quantities are formed from the base ones by applying the rules of multiplication, division and exponentiation.

In 1980, International Committee for Weights and Measures (CIPM) adopted Recommendation 1 [1], in which plane and solid angles were declared as dimensionless derived quantities, and a class of supplementary units was interpreted as the class of dimensionless derived units. A plane angle was defined as a ratio of two quantities having the same dimension of length. A solid angle was defined as a ratio of an area to the square of length. In 1985, this decision was officially included in the 5th edition of the SI Brochure [2]. In 1995, the 20th CGPM in its Resolution 8 [3] eliminated the class of supplementary units from the SI and declared the radian and steradian to be dimensionless derived units.

As a result of these definitions the units of plane and solid angles in the SI Brochure [4] are currently defined as folows:

$$1\ \mathrm{rad} = 1\ \mathrm{m\ m^{-1}}, \qquad 1\ \mathrm{sr} = 1\ \mathrm{m^2\ m^{-2}}. \tag{1}$$

The names radian and steradian can be applied (but not necessarily) for the convenience of distinguishing the dimensionless derived units of the plane and solid angles.

To measure the value of a plane angle, another unit is also used—a degree. The degree is not an SI unit, but is allowed to be used on a par with the radian. And in the SI Brochure [4] it is defined by the expression

$$1^\circ = (\pi/180) \text{ rad.} \tag{2}$$

It follows that the degree also turns out to be a dimensionless number. These definitions remained unchanged in the new edition of the SI Brochure [5].

The decision of the 20th CGPM to interpret the radian and the steradian as dimensionless derived units and eliminate the class of supplementary units as a separate class in the SI was based on the Recommendation U1 of the Consultative Committee for Units (CCU) (1980) [6] and the 1980 CIPM Recommendation 1 [1].

The contradictions between the intuitive conception of the angles as dimensional quantities and the mathematical relations accepted for the defining of the angles as dimensionless quantities have led to the appearance of a large number of the papers analyzing this case. Some authors [7, 8] proposed to change the relation for the plane angle, introducing into it a dimensional coefficient. In the papers by other authors [8–12], it was proposed to consider the plane angle as dimensional and refer it to the base quantities, and its unit, the radian, to the base SI units. In [13–18], the difficulties of the agreement of the non-dimensional status of angles and the existing equations of mathematics and physics are discussed. Quincey and his colleagues [19–21] analyzed various versions of the treatment of angles and concluded that so far it is not possible to eliminate all the contradictions associated with the current status of the angles in the SI. In these works, even an assumption was made that in order to resolve these contradictions it was necessary to change the basic equations of physics and mathematics.

The basic statements of 1980 CIPM Recommendation 1 [1] and their consequences are examined in this article. Section 1 analyzes the arguments underlying the transferring of angles into the class of dimensionless derived quantities and declaring their units, the radian and the steradian, as the dimensionless number ‘one’. In this section, it is shown that the definition of the angles, as dimensionless quantities, has been made because of the incorrect interpretation of the mathematical relations that connect the value of a plane angle with the length of the circular arc bounded by it, and its radius, and the solid angle to the area of the part of the sphere bounded by this angle, and its radius. Another point that may arise confusion in the definition of the dimensionality of the plane angle is the existence of two types of trigonometric functions. Section 2 discusses inconsistencies and contradictions arising in the wording of the SI units as a result of this change in the status of angles. Section 3 establishes the relationship between the plane and solid angles and their units.

1. The analysis of the arguments for declaring angles as dimensionless derived quantities

In the 1980 CIPM Recommendation 1 [1], the main driving motives of the decision to declare plane and solid angles by dimensionless derived quantities were formulated. As can be seen from the text of this Recommendation the main arguments for such a desision can be formulated as the following two assertions:

- that the plane angle is expressed as the ratio of two lengths and the solid angle is expressed as the ratio of the area to the square length,
- that the present structure of the SI with seven base units is the only possible coherent system convenient for use, and there are no coherent and at the same time convenient systems containing the plane and solid angles as base quantities.

The first assertion is the only justification for transferring plane and solid angles into the class of dimensionless quantities. The second assertion serves to justify the declaring of both these angles as derived quantities.

As to the second assertion, it is not very convincing. First, it is not necessary that both angles be introduced into the same class. The question of the introducing of any quantity into the base class should be decided by analyzing the physical nature of the quantity and its relationship with other SI quantities. And the structure with the seven base quantities is not absolute and unchanged. After all, there had been already by that time an experience of changing the structure of the SI. In 1971, there was a precedent of expanding a list of base units from six to seven—the amount-of-substance unit was introduced into the SI as a base unit, and not a derived one. And this neither caused inconvenience of work, nor broke the coherence of the system of units.

Let us consider the first assertion in more details. We start with a plane angle and try to examine what the formula, connecting the angle and two lengths, expresses. To this end, we solve the problem of determining the length l of an arc of the radius r, bounded by the central angle φ. Figure 1 shows the arc and the corresponding central angle. To solve this problem, the arc is supplemented to a circle of the same radius. We call the angle corresponding to the full turn of one side relative to the other the full plane angle and denote it by Φ[1].

From figure 1 it can easily be seen that the ratio of the length l of the arc to the length of the entire circle $2\pi r$ is equal to the ratio of the angle value φ to the full plane angle value Φ. We can write this equation as

$$\frac{\varphi}{\Phi} = \frac{l}{2\pi r}. \tag{3}$$

In metrology there is a special form of writing any quantity, proposed by Maxwell [22], $\varphi = \{\varphi\}[\varphi]$. Here $[\varphi]$ is the unit of measurement for φ, and $\{\varphi\}$ is the numerical value (dimensionless number) of the quntity φ measured in the unit $[\varphi]$. Using this form of recording angles in the left-hand side of the equation (3), we can rewrite it as

$$\frac{\{\varphi\}[\varphi]}{\{\Phi\}[\varphi]} = \frac{l}{2\pi r}.$$

Here the units $[\varphi]$ in the left-hand side of the equation are simplified leaving the ratio of the two dimensionless numbers.

[1] The full angle Φ is sometimes used as a unit of a plane angle. This unit is called the revolution (symbol ‘rev’).

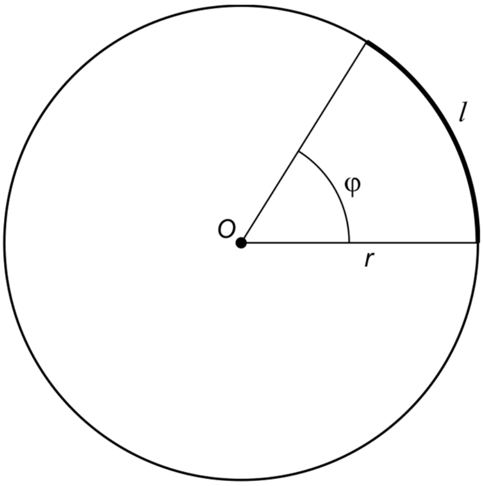

Figure 1. The illustration for calculating the ratio between quantities l, r, and φ.

Rewriting the resulting equality, we shall have an expression for $\{\varphi\}$

$$\{\varphi\} = \frac{\{\Phi\}}{2\pi} \cdot \frac{l}{r}. \tag{4}$$

It connects the numerical value of the plane angle with the arc length and its radius. However, this ratio does not determine either the dimension of the angle or its unit. This unit should be defined by a certain angle, the size of which is chosen for reasons of the convenience of use.

Depending on the choice of the unit $[\varphi]$ of the plane angle, the ratio will be presented in different forms. If we measure angles in degrees, then $[\varphi] = 1°$. In this case, the dimensionless number $\{\Phi\}$ is equal to 360, and the formula (4) takes the form

$$\{\varphi\} = \frac{180}{\pi} \cdot \frac{l}{r}. \tag{5}$$

This coefficient $180/\pi$ (in general $\{\Phi\}/2\pi$) arises in mathematical calculations related to the angles and functions for them, violating the compactness of mathematical formulas. And in doing many calculations, it will repeatedly occur, making calculations too big and cumbersome.

If we choose such a unit of the plane angle that the dimensionless number $\{\Phi\}$ is equal to 2π, then the formula (4) takes the compact form[2]

$$\{\varphi\} = \frac{l}{r}. \tag{6}$$

The corresponding unit of the plane angle, ensuring the equality $\{\Phi\} = 2\pi$, is called the radian (symbol 'rad'). And the radian itself is defined based on the condition that the full plane angle is equal to

$$\Phi = 2\pi \text{ rad}. \tag{7}$$

The expression (6) shows that the ratio of the two lengths determines not the angle φ itself, but only its numerical value, in radian units. This value $\{\varphi\}$ is indeed a dimensionless number by definition. But, contrary to the statement in CIPM Recommendation 1 (1980), the expression (6) does not produce any restrictions on the dimension of the angle itself. Also, there are no other expressions leading to the conclusion that the plane angle is dimensionless quantity.

These arguments are also true for the solid angle ω, for which it is easy to derive an expression similar to the relation (4) for a plane angle

$$\{\omega\} = \frac{\{\Omega\}}{4\pi} \cdot \frac{S}{r^2}, \tag{8}$$

where $\{\omega\}$—the numerical value of the solid angle under consideration in units of $[\omega]$, $\{\Omega\}$—the numerical value of the full solid angle in the same units, S—the area of surface bounded by the solid angle ω on a sphere of radius r centered at the angle vertex.

The unit of solid angle steradian (symbol 'sr') is chosen, by analogy with the unit of a plane angle, so that the expression for the numerical value of the angle $\{\omega\}$ has a compact form of a ratio of the area S to the square radius

$$\{\omega\} = S/r^2. \tag{9}$$

So, the ratio of an area to the square length determines not the solid angle itself, but the numerical value $\{\omega\}$ of the solid angle ω, measured in steradians. The steradian can be defined in the same way as the radian by setting the value of the full solid angle

$$\Omega = 4\pi \text{ sr}. \tag{10}$$

Let us indicate one more factor that affects the determination of the values of angles and their units—the effects of the general relativity (GR). In GR, space is not flat, but curved. Many properties of geometric objects in a curved space differ significantly from their properties in a flat Euclidean space. For example, the sum of the angles of a triangle is not equal to π rad, as is the case with the ordinary geometry on a plane. And even the refined formulas (4) and (8) in a curved space are no longer true. In general, they cannot be used to determine the values of angles. For example, in astronomical observations and calculations, the effects of GR are quite significant and these formulas can give a large error in the calculations. It is necessary to use the formulas of non-Euclidean geometry.

In a curved space, the length of the arc $l(r, \varphi)$ and the area of the segment of the sphere $S(r, \omega)$ are not linear functions of r and r^2, respectively. However, in small spatial domains, the deviations of the formulas of non-Euclidean geometry from the geometry of a flat space are small. Therefore, with small r, the expressions connecting the length, area, radius, and numerical values of the angles will differ little from the formulas (4) and (8). And the smaller the value of r, the more accurate these relations will be. In the limit of $r \to 0$, they become exact expressions

$$\{\varphi\} = \lim_{r\to 0} \frac{\{\Phi\}}{2\pi} \frac{l}{r}, \qquad \{\omega\} = \lim_{r\to 0} \frac{\{\Omega\}}{4\pi} \frac{S}{r^2}. \tag{11}$$

It is clear that definitions of the radian and the steradian in terms of the arc length and surface area in a curved space are

[2] The formulas similar to (5) and (6) are given in the Mathematical Encyclopedia [23, p 15], article 'Circle', to express the length of the arc through the radius and angle value.

incorrect for finite values of radius r, whereas the relations (7) and (10) do not depend on lengths at all and are valid in spaces of any curvature.

There is another point that may arise confusion the issue of dimensional character of angles—trigonometric functions. For example, in [13, 19] a series expansion of a trigonometric function according to the powers of its argument is considered as strong evidence of the dimensionless character of the plane angle. Such confusion arises because the argument of the trigonometric functions is usually considered only angles [17]. But it is not so. In mathematics there exist two types of trigonometric functions. Functions of angles and functions of dimensionless numbers. The former were introduced more than two thousand years ago as the ratio of the lengths of the sides of a right triangle. These functions were defined in a small region of angles from 0 to $\pi/2$ radians.

The following statement is proved in mathematics [24]: 'On the whole real number axis $-\infty < x < \infty$, there are two unequivocally defined functions $S(x)$ and $C(x)$ that satisfy certain conditions. The conditions for the existence of these functions are chosen in such a way that they are satisfied for the trigonometric functions of the angles $\sin\varphi = \sin(x\ \text{rad})$ and $\cos\varphi = \cos(x\ \text{rad})$. Then in the interval of $0 \leqslant x \leqslant \pi/2$, the functions of the angles will fully coincide with the functions of dimensionless arguments $\sin(x\ \text{rad}) = S(x)$ and $\cos(x\ \text{rad}) = C(x)$.

By extending these equalities to all values of dimensionless numbers x, mathematicians extended the definition domain of the trigonometric functions of angles φ to the entire infinite range of their values $-\infty\ \text{rad} < \varphi < \infty\ \text{rad}$. Moreover, the functions $S(x)$ and $C(x)$ were also called trigonometric functions and denoted by the same symbols as the trigonometric functions of angles: sin and cos. But we must always remember that the trigonometric functions of angles and the trigonometric functions of dimensionless numbers are, strictly speaking, completely different functions. They have different definition domains. This was indicated by Brownstein [10]. In order to avoid confusion, he proposed to designate differently these two types of functions. The functions of dimensionless arguments are still denoted as sin and cos, and the functions of angles as Sin and Cos.

For these two types of functions, all trigonometric formulas have exactly, the same form. But when differentiation, or integration, or a series expansion, there immediately appear differences between them. All equations in physics and mathematics are written using only the trigonometric functions of dimensionless arguments. All mathematical reference books are compiled for trigonometric functions of dimensionless arguments (not angles). According to the formulas of a series expansion of functions of one type, no conclusion can be made about the dimensionality of the arguments of functions of another type. The dimensionality of a plane angle does not depend on the series expansion of the trigonometric function of a dimensionless argument. The series expansion of a trigonometric function of an angle is carried out not by its argument (angle value) but by the numerical value of an angle expressed in radians (see, for example, [14]).

2. The consequences of transferring angles to the class of dimensionless derived quantities

As a result of the 1995 reform, some inconsistencies and contradictions appeared in the formulations of the SI Brochure. The transfer of the radian and steradian to the class of dimensionless derived units led to the appearance of the section 'Units for dimensionless quantities, also called quantities of dimension one' in the SI Brochure [4][3]. The dimensionless quantities are in fact divided into three classes there.

The first class includes dimensionless quantities that are obtained as a result of certain combinations of dimensional physical quantities. The unit of such a dimensionless quantity is the number one, which is called in the Brochure 'a dimensionless derived unit'. At the same time, the Brochure does not specify which base SI units this 'derived unit' is produced from.

The second class of dimensionless quantities given in the SI Brochure is the numbers that represent counting of objects: a number of molecules, a degree of degeneracy of the quantum level, and so on. The unit of these dimensionless quantities is also the dimensionless number one. However, this dimensionless number one is no longer a derived unit in the SI Brochure, but instead is treated as 'a further base unit' [4]. And this contradicts the assertion about the need to adhere to a rigid scheme with seven base units in the SI.

And finally, the third class of dimensionless quantities includes the plane and solid angles mentioned above. The following is said about them in this section of the SI Brochure [4]: 'In a few cases, however, a special name is given to the unit one, in order to facilitate the identification of the quantity under consideration. This is the case with the radian and the steradian. The radian and the steradian have been identified by the CGPM as special names for the coherent derived unit one, to be used to express values of the plane angle and the solid angle, respectively, and are therefore included in Table 3'.

And what is it—the dimensionless number one, which is marked with two different names for two different quantities to distinguish what quantity it refers to? This is, after all, two different units for two different quantities. Such interpretation of this text in the SI Brochure is in complete agreement with the fact that the plane angle and the solid angle are quite different quantities, the quantities of different kinds. The plane angle is a two-dimensional (2D) geometric object on a plane, and the solid angle is a three-dimensional (3D) geometric object in a 3D Euclidian space. These quantities cannot be compared. And they cannot be added or subtracted either[4]. Comparing the plane angle and the solid angle is just like comparing the length and the area. And there was no need to define them as equally dimensionless quantities, immediately introducing

[3] In the new edition of the SI Brochure [5], adopted at the 26th CGPM, this section is excluded.

[4] This circumstance provides an effective way to control the correctness of mathematical calculations. If the dimensions of the terms in the equation under study turn out to be different, it means that there was an error somewhere earlier in mathematical transformations.

different units of their measurement, without giving a practical definition for these units at that.

The definitions of the radian and the steradian as derived units in the SI Brochure [4] also have some internal contradiction. According to the definitions given there, the two derived units, the radian and the steradian, are expressed in terms of only one base unit meter by the relations (1). If the usual algebraic rules are applied to these formulas, then the meter in the numerator and denominator is reduced. As a consequence, the definitions of the units of angles will not contain any base SI units at all. And if an equation does not have a quantity, then in this equation nothing depends on the missing quantity. Therefore, the radian and the steradian are not defined by any base SI unit. And this contradicts the basic concept of the coherent system, according to which all derived units are determined coherently through the base ones. So, the assumption that the radian and the steradian (1) are derived from the current base SI units contradicts itself.

Another problem related to the change in the status of angles in the SI is the use of the dimensionless units for measurements. Measurement is a process of physical manipulations with at least two material objects—the angle being measured and its unit of measurement. To perform any physical actions with the dimensionless number is not possible. A dimensionless number is a mathematical concept, an abstraction. It is impossible to measure angles with the dimensionless number one. There are no other definitions of the units of angles in the current SI Brochure [4].

In practice, the radian and the steradian are defined as the plane angle and the solid angle of certain sizes. It is these angles, and not the dimensionless number one, that are used as units of measurement for the plane and solid angles. The measurement methods for these angles can use both the length of the arc, or the area of spherical surface. But the units of angles themselves are not defined through lengths and areas.

In the 9th edition of the SI Brochure [5], adopted in November 2018 at the 26th CGPM, certain changes were made to the definitions associated with the units of angles. In the new edition the definitions themselves of the radian and the steradian remained the same as in the previous edition—the dimensionless number one. While the explanatory footnotes to these units definitions in the 8th and 9th editions are strikingly different. In the footnote of the 8th edition of the SI Brochure, the radian and steradian are simply specific names for a dimensionless number one. Whereas in the footnotes of the 9th edition they are already angles of a certain size. These footnotes clearly indicate that the radian and steradian are not a dimensionless number one in this edition of the SI Brochure, as written several paragraphs above.

3. The relationships between plane and solid angles and their units

In order to determine the status of plane and solid angles and their units in the SI, it is necessary to investigate the geometric nature of these quantities and the relationship between them.

The plane angle is defined as a geometric figure consisting of two different rays starting from a single point [23]. More precisely, the angle represents the entire region of the plane enclosed between these two rays. The rays are called the sides and their common point is called the vertex of the angle.

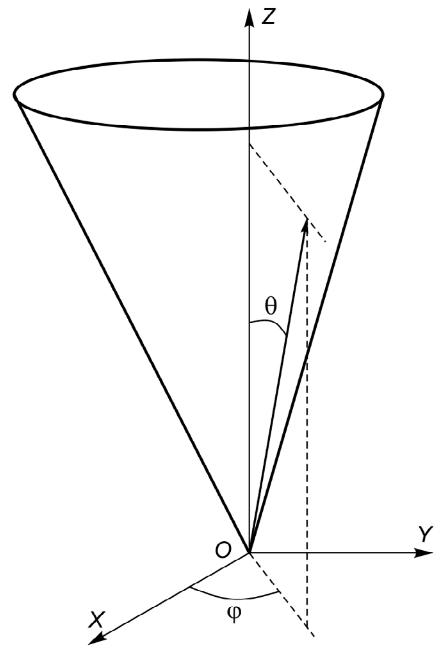

Figure 2. Solid angle and assotiated coordinate systems.

The value of a plane angle is determined by the magnitude of the deviation of one ray direction from another. This deviation does not depend on any lengths. In the paper [11] it is shown that none of the angle definitions refers to any other quantity. In fact, a plane angle is a local object [14] consisting of one point (vertex of the angle) and two directions given at this point, which have no length. It means that the plane angle is an independent quantity in the SI with its own dimension, which is in no way connected with the dimensions of any other SI quantities. So, it cannot be a derived quantity in the SI.

The property of a plane angle to characterize the value of deviation of one ray from another is used in mathematics to build a polar coordinate system on a plane, as well as cylindrical, spherical, and other kinds of coordinate systems in 3D space. We will use this point to analyze the relationship between the plane and solid angles.

In [25], the solid angle is defined as part of space bounded by one cavity of a conical surface (see figure 2). As in the case of the plane angle, the lengths of the rays that make up the conical surface of the solid angle do not matter. The spatial direction of these rays is what is of importance. In contrast to the plane angle, the solid angle cannot be set up on a plane. It is a 3D object. The conical surface itself is a continuous closed set of rays starting from one point—the vertex of the solid angle.

It is almost obvious that the solid angle is formed by a set of plane angles. Figure 2 shows the solid angle ω and the associated Cartesian and spherical coordinate systems.

The direction of any ray starting from the origin of the coordinates is defined in the spherical coordinate system by two plane angles θ and φ, as shown in figure 2. These angles can take values from intervals $0 \leqslant \varphi < \Phi$, $0 \leqslant \theta \leqslant \Phi/2$. For each value of the angle φ, the corresponding ray of the conical surface will form a certain plane angle θ with the axis OZ. Changing the value of the angle φ from zero to Φ, we get a set of plane angles $\theta(\varphi)$ that fill the entire solid angle under consideration. This process is similar to the process of formation of a flat 2D figure by a set of straight line segments or a 3D object by a set of 2D flat figures. This means that the solid angle is a derived quantity in the SI formed by plane angles, just as the area is a derived quantity formed by lengths. It follows that the coherent unit of the solid angle in the SI should be equal to rad^2.

It is not difficult to find the connection between the steradian and the radian. The whole set of directions of the rays defining the conical surface of the solid angle will be determined by the function $\theta(\varphi)$. The value of the solid angle ω is obtained by integrating the element of the solid angle $d\omega = \sin\theta d\theta d\varphi$ over the region $0 \leqslant \varphi \leqslant \Phi$, $0 \leqslant \theta \leqslant \theta(\varphi)$

$$\omega = \int_0^{\Phi} \int_0^{\theta(\varphi)} \sin\theta \mathrm{d}\theta \mathrm{d}\varphi. \tag{12}$$

Integration over the angle θ is easily performed through transfer to the dimensionless integrating variable s using the formula $\theta = s$ rad $= s\Phi/2\pi$ and connecting the trigonometric functions of the angle and the dimensionless variable from section 1. As a result, the expression (12) takes the form

$$\omega = \frac{\Phi}{2\pi} \int_0^{\Phi} [1 - \cos\theta(\varphi)] \mathrm{d}\varphi. \tag{13}$$

Using this expression, we can find the value of the full solid angle Ω. Let us choose the cavity inside the circular cone as the solid angle ω. The corresponding function $\theta(\varphi)$ will be the constant θ, independent of φ. The integral in (13) gives the following value of the solid angle

$$\omega = \frac{\Phi^2}{2\pi}(1 - \cos\theta). \tag{14}$$

At $\theta = 0$ (the conical surface degenerates into the axis OZ) the corresponding solid angle will be zero. The full solid angle Ω is obtained at the maximum value of the angle $\theta = \Phi/2$, for which $\cos(\Phi/2) = -1$. The expression (14) gives the following value for Ω

$$\Omega = \Phi^2/\pi. \tag{15}$$

Comparing this expression with the formula (10), we obtain the expression for the value of the steradian, provided the unit of the plane angle is set arbitrarily

$$1 \text{ sr} = \frac{\Phi^2}{4\pi^2}. \tag{16}$$

If a radian measure is used for the plane angle, this expression takes the following form

$$1 \text{ sr} = 1 \text{ rad}^2. \tag{17}$$

This means that the steradian is a coherent derived unit of the radian, but not of the meter. The unit of the plane angle—the radian does not depend on the other SI units and is set axiomatically, like all other base SI units. The other unit of plane angle, the degree, is still defined by formula (2). But it is no longer a dimensionless number, but has the dimension of a plane angle, like the radian.

Some of the results discussed in this work were published in 2018 in the electronic preprint Arxiv.org [26]. Regretfully, in the first two versions of the preprint (dated October 29 and November 6), there occurred an error in deriving the relation connecting a solid angle with plane angles. In the third version of the preprint (dated November 8) this error was already corrected.

4. Conclusion

The findings of the studies carried out in this paper are as follows.

1. There are no grounds for considering the plane and solid angles dimensionless quantities. Mathematical relations connecting plane and solid angles with lengths and areas were mistaken for definitions of angles. In fact, these relations determine only the numerical values of the plane and solid angles in units of radians and steradians. The units of angles themselves cannot be obtained from these relations. They must be defined for other reasons.
2. Formulas for the series expansion of trigonometric functions are often incorrectly used to justify the dimensionless character of a plane angle. In mathematics there are two types of trigonometric functions: trigonometric functions of angles and trigonometric functions of dimensionless numbers. Their series expansions differ from each other. From the formula for the series expansion of a trigonometric function of a dimensionless argument it is impossible to conclude about the dimensionality of a plane angle.
3. The plane and solid angles are quantities of different kinds, having different geometric dimensions. They cannot be compared with each other in magnitude. Therefore they have different units of measurement, the radian and the steradian, not the dimensionless number one.
4. The plane angle is a quantity independent of other quantities. It should be added to the base quantities, and its unit, the radian, to the base units of the SI.
5. The units of the radian and the steradian can be defined by fixing the exact values of the full plane Φ and solid Ω angles:

- the radian is defined on the condition that the full plane angle is equal to $\Phi = 2\pi$ rad,
- the steradian is defined on the condition that the full solid angle is equal to $\Omega = 4\pi$ sr.

These definitions are independent of lengths. They are also true when considering the effects of GR.

6. The solid angle is a derived quantity of the plane angle, not of the length. Its coherent unit is the steradian, which is equal to the squared radian.

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