MEMORANDUM

To: ALCON

From: Peter Fisher

Subject: A simple derivation of Unruh Radiation

Date: October 11, 2023

Fig. 1 shows a PARTICLE frame containing electromagnetic radiation of frequency ω' and wave number $\vec{k} = \pm\omega'\hat{x}$. There is a set of PARTICLE frames, each corresponding to a different time. In each PARTICLE frame, the particle is at rest, $u' = 0$, and accelerating with $d\beta_{u'}/dt' = a$. The time dependent velocity v for the Lorentz transformation between the two frames is chosen so that the particle remains at rest and accelerates with a.

Our first task is to find $\beta_v(t)$. Next, we will find ω and $\vec{E}$ in the LAB frame. Finally, we Fourier transform $\vec{E}$ and find that the power spectrum follows a black body spectrum and compute the temperature T.

1. The velocity of the particle in the LAB frame is

$$u = \frac{v + u'}{1 + vu'/c^2}$$

or

$$\beta_u = \frac{\beta_v + \beta_{u'}}{1 + \beta_v\beta_{u'}}.$$

We need to find how $\beta_u$evolves with time as a function of acceleration and time *only*, which means finding $d\beta_u/dt'$ and setting it equal to $d\beta_v/dt'$ to ensure the particle remains at rest in all PARTICLE frames.

$$\frac{d\beta_u}{dt'} = \frac{(d\beta_v/dt' + d\beta_{u'}/dt')(1 + \beta_v\beta_{u'})}{(1 + \beta_v\beta_{u'})^2} \quad (1)$$

$$-\frac{(\beta_v + \beta_{u'})(\beta_{u'}d\beta_v/dt' + \beta_v d\beta_{u'}/dt')}{(1 + \beta_v\beta_{u'})^2} \quad (2)$$

$$= \frac{a}{\gamma_v^2}. \quad (3)$$

In any PARTICLE frame, $\beta_{u'} = 0$ and $d\beta_{u'}/dt' = a$. The last step comes from applying these conditions on the particle in the PARTICLE frame. t' is the proper time of the PARTICLE frame.

2. The requirement that the particle remain at rest and accelerate at a in all PARTICLE frames gives $\beta_u = \beta_v$

$$\frac{d\beta_v}{dt'} = \frac{d\beta_u}{dt'} = \frac{a}{\gamma_v^2}$$

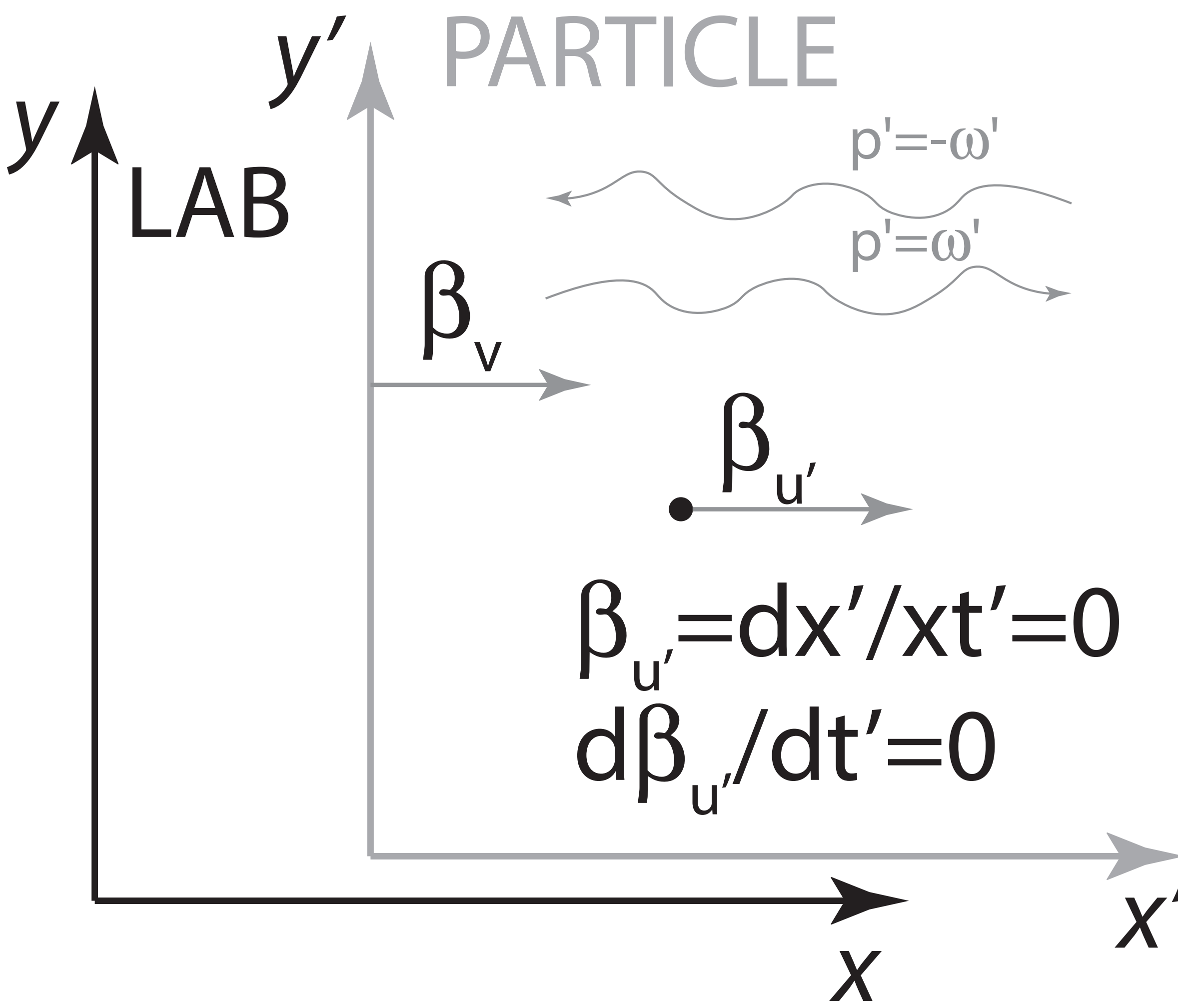

Figure 1: PARTICLE and LAB frames.

and

$$\begin{aligned}
\int \frac{d\beta_v}{1-\beta_v^2} &= \int a dt' && (4) \\
-\frac{1}{2}\ln(1-\beta_v) + \frac{1}{2}\ln(1+\beta_v) &= at' + C && (5) \\
\ln\frac{1+\beta_v}{1-\beta_v} &= 2at' + C && (6) \\
\frac{1+\beta_v}{1-\beta_v} &= C\exp\left(2at'\right). && (7)
\end{aligned}$$

We fix C by saying that the particle "turns around", $\beta_v = 0$, at $t' = 0$, so $C = 1$. Solving Eq. 7 gives $\beta_v = \tanh at'$. Also, $\gamma_v = \cosh at'$ and $\beta_v \gamma_v = \sinh at'$.

3. Next, we transform the electromagnetic wave from the PARTICLE frame to the LAB frame,

$$\begin{aligned}
\omega &= \gamma_v \omega' \pm \beta_v \gamma_v k_x && (8) \\
&= \omega' \left(\cosh at' \pm \sinh at'\right) && (9) \\
&= \omega' e^{\pm t'}. && (10)
\end{aligned}$$

The positive solutions correspond to photons moving in the $\hat{x}'$ direction and the negative solutions correspond to photons moving in the $-\hat{x}'$ direction. At each time t, the PARTICLE and LAB frames are connected by a Lorentz transformation with $\beta_v(t)$, a photon in the PARTICLE frame moving in the $\hat{x}'$ also moves in the $\hat{x}$ direction in the LAB frame, and so on.

4. The electric field is $\vec{E} \propto e^{i\phi(t')}$ and the phase is

$$\begin{aligned}
\phi\left(t'\right) &= \int_{-\infty}^{t'} dt'' \omega' e^{at''} && (11) \\
&= \pm\frac{\omega}{a} e^{\pm at'} |_{-\infty}^{t'}. && (12)
\end{aligned}$$

The positive solution converges, the negative does not. For the positive solution, we have

$$\phi\left(t'\right) = \frac{\omega'}{a} e^{at'}.$$

5. The power density is proportional to the norm of the Fourier transform of the electric field

$$P(\Omega) = \left|\int_{-\infty}^{\infty} dt' e^{i\Omega t'} e^{i(\omega'/a)e^{at'}}\right|^2. \quad (13)$$

Carrying out the integral is straight-forward, but tedious: substitute $y = \exp at'$ then use

$$\begin{aligned}
\int_0^{\infty} x^{\mu-1} \sin ax dx &= \frac{\Gamma(\mu)}{a^{\mu}} \sin\frac{\mu\pi}{2} && (14) \\
\int_0^{\infty} x^{\mu-1} \cos ax dx &= \frac{\Gamma(\mu)}{a^{\mu}} \cos\frac{\mu\pi}{2}. && (15)
\end{aligned}$$

Then

$$\int_{-\infty}^{\infty} dt' e^{i\Omega t'} e^{\omega'/ae^{at'}} = \frac{1}{a}\Gamma\left(\frac{i\Omega}{a}\right)\left(\frac{\omega'}{a}\right)^{i\Omega/a} e^{-\pi\Omega a}. \tag{16}$$

Squaring and using

$$\left|\Gamma\left(\frac{i\Omega}{a}\right)\right|^2 = \frac{\pi a}{\Omega}\sinh\frac{\pi\Omega}{a} \tag{17}$$

gives

$$\left|\int_{-\infty}^{\infty} dt' e^{i\Omega t'} e^{\omega'/ae^{at'}}\right|^2 = \frac{2\pi}{\Omega a}\frac{1}{e^{2\pi\Omega a-1}}.$$

Comparing with the black body spectrum gives $T = \hbar a/2\pi k_B$.

1 Some observations and questions

This derivation of the black body spectrum resulting from a single, arbitrary frequency in an accelerating frame seems miraculous: how can just special relativity and electricity develop the black body spectrum? In fact, using $E = \hbar\omega$ brings in quantum mechanics as this resulted from study of the black body spectrum to begin with.

The dependence on ω' disappears in Eq. 17. The resulting spectrum should be independent of frequency since we could start from a third frame moving at constant velocity with respect to the PARTICLE frame. The net effect of doing this is changing the turn-around time, which in turn changes the value of the integration constant C and $\beta_v = \tanh a\,(t' - t'_{turn})$, ultimately given a black body spectrum in the LAB frame.

What we have shown is that any radiation in an accelerating from appears as a black body spectrum in the LAB frame, a different statement than saying an accelerating object radiates a black body spectrum. The "object" that appears in Fig. 1 serves no purpose, the EM radiation gives us the black body spectrum.