JASON

MEMORANDUM

To: Kris

From: Peter

Subject: Memo #90a: The Intercept Problem (revised)

Date: August 18, 2024

This note considers the intercept problem - how to change the velocity of an intercept vehicle to collide with a target vehicle[1] based on the maneuvers of the target. Bounding the needed acceleration of the target vehicle based on the target's acceleration capability serves as this memo's only goal - the method probably does not provide an optimal solution, does not take into account prior knowledge of the target, and does not advance beyond two dimensions. Obtaining the result proved difficult, and the Appendix provides the algebraic manipulations for checking.

Figure 1 frames the problem: a target located at $\vec{r}_1'$ moves with velocity $\vec{u}_1$ at time t_1 while an interceptor starts at $\vec{r}_1$ with velocity $\vec{v}_1$ on an intercept course such that,

$$\vec{R}_1 = \vec{r}_1' - \vec{r} = \vec{v}_{c,1} t_{go} \tag{1}$$

where $t_{go} = t_{intercept} - t_1$ gives the time to intercept. Note the minus sign in Eq. 1 arises from $\dot{\vec{R}}_3 = \dot{\vec{r}}_1' - \dot{\vec{r}}_1$ - the vector connecting the two vehicles must decrease with time. The vector connecting the two vehicles $\vec{R}_1$ must lie parallel to their relative velocity $\vec{v}_{c,1}$ with $\vec{R}_1 \cdot \vec{v}_{c,1} > 0$ and

$$t_{go} = \frac{|\vec{R}|}{|\vec{v}_c|}$$

for the intercept to take place. Restated, $\vec{R} \times \vec{v}_c = 0$ serves as the intercept condition used to determine the interceptor's response to a maneuver by the target.

At t_1, the target maneuvers by applying an acceleration $\vec{a}'\tau = \Delta\vec{u}$, where τ gives the characteristic response time of the interceptor: at between t_1 and $t_2 = t_1 + \tau$, the interceptor measures the change in velocity of the target, computes a response, and, at t_2, initiates an acceleration $\vec{a}_1$.

The maneuver of the target breaks the intercept condition $\vec{R}_2 \times \vec{v}_{c,2} = 0$ and the target seeks to restore this condition by the time $t_3 = t_1 + 2\tau$ by matching the target's velocity change $\Delta\vec{u}_1$ and making up the distance $\Delta\vec{u}_1\tau/2$ it lost during the interval $t_1 - t_2$. The following outlines the calculation of $\vec{a}_1 = \Delta\vec{v}_1/\tau$ needed to restore the intercept condition. The Appendix gives the line-by-line calculation, and Figure 1 illustrates the problem and solution graphically.

[1] This note will refer to the target vehicle as the target and the interceptor vehicle as the interceptor.

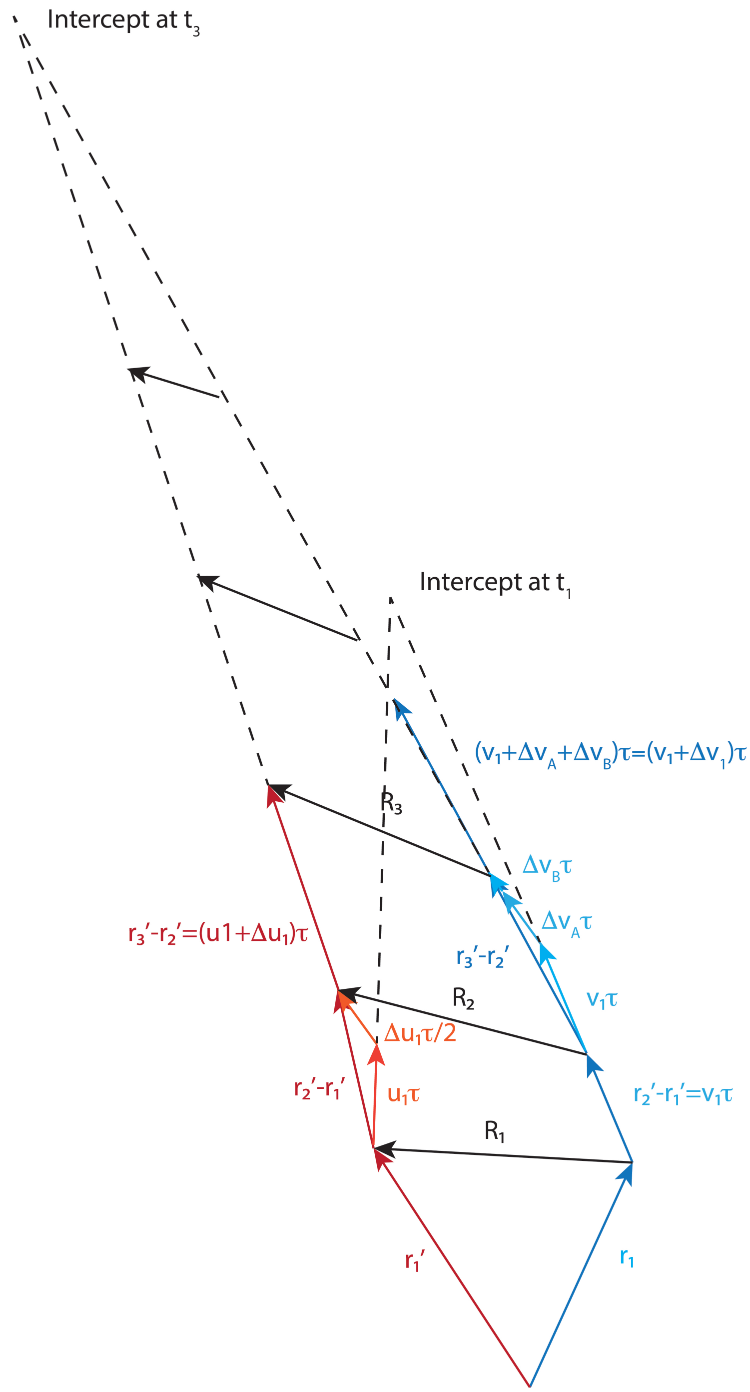

Figure 1: Framing diagram for the intercept problem. The red arrows on the left show the target's trajectory, and the blue on the right shows the trajectory of the interceptor.

Eq. 5-10 shows the positions, velocities, and times at t_1. t_{go} gives the time until the intercept occurs and decreases at t increases. Eq. 11-18 gives the positions and velocities at $t_2 = t_1 + \tau$ after the target stops accelerating - at this point, the intercept condition no longer holds, and the target must accelerate to restore it. Note that t_{go} decreases with time.

Eq. 19-31 gives the positions and velocities at t_3 and Eq. 24-25 show the decomposition of the response velocity $\Delta\vec{v}_1$ into a piece that matches the interceptor's change in velocity and a piece that makes up for the distance lost in the interval $t_1 - t_2$. This section results in expressions for $\vec{R}_3$ (Eq. 32) and $\vec{v}_{c,3}(Eq.\ 31)$. Evaluating in Eq. 32-41$\vec{R}_3 \times \vec{v}_{c,3} = 0$ in terms of the unknown $\vec{v}_B$ - the additional target velocity needed to make up distance lost, giving a general intercept expression.

$$\boxed{\Delta\vec{v}_B \times \left(\vec{R}_1 + \Delta\vec{u}_1\tau - \frac{3}{2}\vec{v}_{c,1}\tau\right) - \Delta\vec{u}_1 \times \vec{v}_{c,1}\tau - \vec{R}_1 \times \vec{v}_{c,1} = 0}. \tag{2}$$

Evaluating Eq. 2 in terms of components gives an affine equation for the components of $\Delta\vec{v}_1$ of the form,

$$\alpha\Delta\vec{v}_{1,x} + \beta\Delta\vec{v}_{1,z} + \gamma = 0$$

Eq. 43, 44, and 45 give expressions for α, β, and γ. Eq. implies an infinite number of solutions. Practicalities will determine the values of the components of $\Delta\vec{v}_1$, such as,

- Upper and lower limits on the vehicle's velocity
- The acceleration that the propulsion system can provide
- The acceleration time
- The desired intercept time.

If $\vec{R}_1 \times \vec{v}_{c,1} = 0$, $\vec{R}_1 = \vec{v}_{c,1}t_{go}$ and Eq. 2 becomes the restricted guidance law,

$$\boxed{\vec{R}_1 \times \Delta\vec{u}_1\frac{\tau}{t_{go}} - \Delta\vec{v}_B \times \left(\vec{R}_1\left(1 - \frac{3\tau}{2t_{go}}\right) - \Delta\vec{u}_1\tau\right) = 0} \tag{3}$$

We now use Eq. 3 to analyze four example engagements. A head-on engagement starts with the target 2 km from the interceptor with a closing velocity of 2 km/s, giving $t_{go} = 1$ s and $\tau = 0.1$ s. The target accelerates laterally at 50 g, and the interceptor must respond. Fig. 2 shows the relationship between the components of $\vec{v}_1$ and the intercept condition $\vec{R}_1 \times \vec{v}_{c,1}$ as a function of $\Delta\vec{v}_{1,x}$. Fig. 3 shows the relationship between the ratio of the interceptor and target. For this example, the minimum ratio $\|a_1/a_1'\| = 1.0002$ gives $t_{go} = 0.82$ s. The graph also shows that longer t_{go} requires the interceptor to decelerate along its direction of travel.

The second example shows a tail-case with the target and interceptor 2 km apart. The vehicles move in the same direction with the target and interceptor moving at 1 km/s. Here, t_{go} becomes infinite. The target accelerates laterally, and the interceptor must respond. The upper left graph of Fig. 3 shows that at the minimum calculated interceptor acceleration ratio of 1.4, $t_{go} = 2004$ s and $t_{go} = 1$s requires an acceleration ratio of 39.

In the next two examples, the interception occurs along perpendicular trajectories with the target traveling along the $\hat{z}$ axis at 1 km/s and starting at $z = -1km$. The interceptor starts at

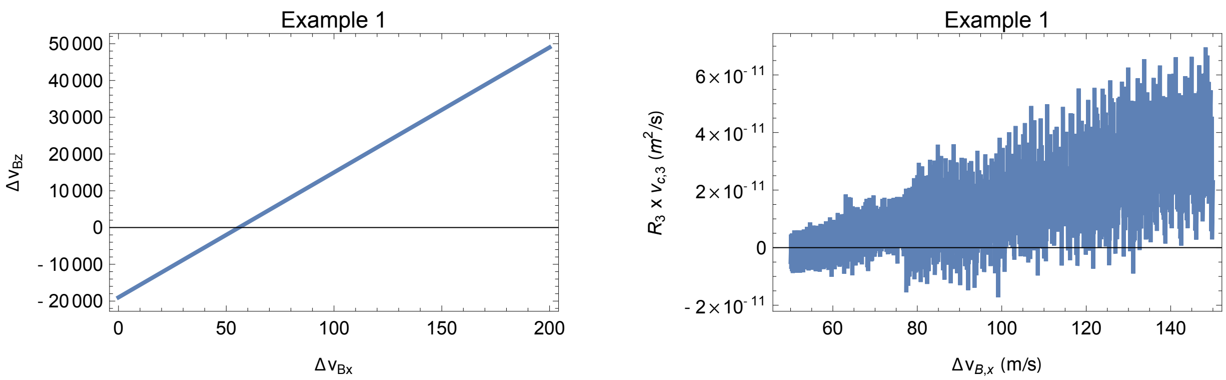

Figure 2: Left: Relationship between components of $\vec{v}_1$. Right: intercept condition $\vec{R}_1 \times \vec{v}_{c,1}$ as a function of $\Delta\vec{v}_{1,x}$.

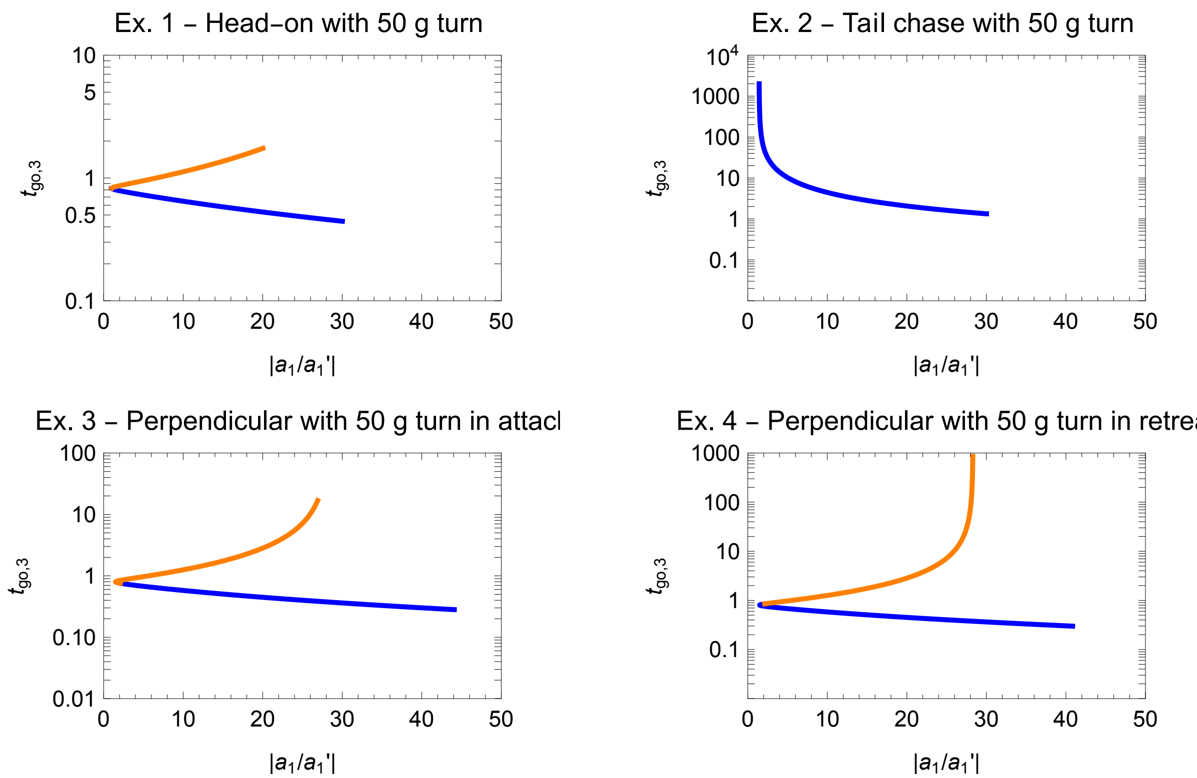

Figure 3: Example engages showing t_{go} as a function of the ratio of the interceptor to target acceleration ratio. The orange portion of the curve indicates the target decelerates along the primary direction of motion, and blue indicates an acceleration along the primary direction of travel. The upper left shows the head-on engagement, the upper right shows a tail chase, the lower left shows the curve for a perpendicular intercept where the target turns toward the interceptor, and the lower right shows the curve for a perpendicular interception where the target turns away from the interceptor.

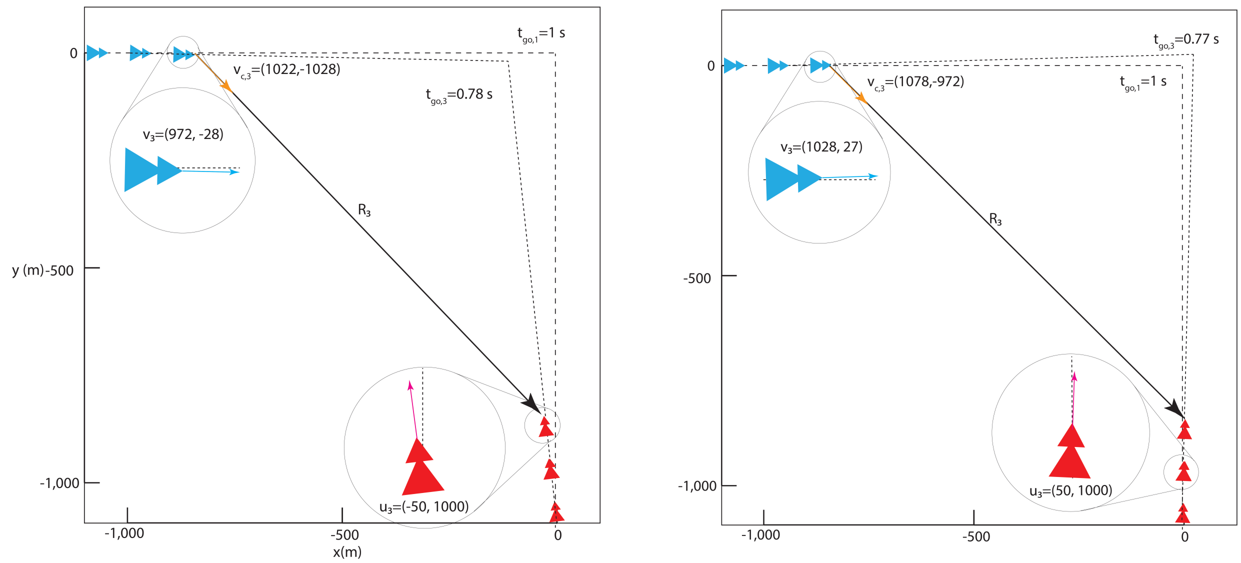

Figure 4: The examples of a perpendicular intercept. The left panel shows the target accelerating toward the interceptor at 50 g, and in the second example, the target turns away from the interceptor at 50 g. The orange portion of the curve indicates cases in which the interceptor decelerated along its primary direction of travel.

$x = -1$ km and travels along the $\hat{x}$ axis at 1 km/s, Fig. 4. $t_{go,1} = 1$s. In the first example, the target turns toward the interceptor with an acceleration of 50 g, and Fig. 3 shows the acceleration ratio versus $t_{go,3}$. The acceleration ratio of 1.5, the minimum, gives $t_{go,3} = 0.78$ s, a shorter time than the 0.8 s had the target not maneuvered.

In the final example, Fig. 4, the target turns away from the interceptor at 50 g, resulting in $t_{go,3} = 0.80$ s for the minimum acceleration ratio of 1.5, Fig 3. The time to intercept turns out slightly longer as the interceptor has to travel slightly further. In both examples, the interceptor decelerates a small amount in its direction of travel for the minimum time to intercept.

In the interval τ before the intercept, the target may make one last maneuver and accrue a displacement $\vec{a}'_{final}\tau^2/2$ to which the interceptor cannot respond. Success will depend on the Goodman number,

$$G = \frac{\vec{a}'_{final}\tau^2/2}{b_i + b_t},$$

where b_t represents the target's projected linear dimension and b_t gives the interceptor's kill radius—either the vehicle's projected linear dimension or the dispersal radius of shrapnel explosively deployed.

As a final example, we consider the case in which the target can respond more quickly than the interceptor. Based on Example 1, the vehicles lie on a collision course t_{go} before the intercept. For interceptor response time τ and target response time τ', $\tau < \tau'$. For the final maneuver of the engagement, the target maneuvers at the last possible instant, so $\tau' = t_{go}$, so if the target begins accelerating with $a'_1\hat{x}$ at $t = 0$, the interceptor can respond at $t = \tau$ and accelerate with $a_1\hat{x}$ and the intercept happens at $t = t_{go}$, Fig. 5. Then,

$$
\left.\begin{array}{rl} x'(t) = & \frac{1}{2}a_1' t^2 \\ x(t) = & \frac{1}{2}a_1 (t-\tau)^2 \end{array}\right\} \rightarrow x(t_{go}) = x'(t_{go}) \rightarrow \frac{a_1}{a_1'} = \left(\frac{t_{go}/\tau}{t_{go}/\tau - 1}\right)^2 \quad (4)
$$

gives the ratio of the accelerations needed. For the nominal case of $\tau' = 5\tau$, $a_1/a_1' = 1.6$. If we take the rule-of-thumb case of $a_1/a_1' = 3$, then $\tau'/\tau = t_{go}/\tau = 2.6$.

The final example shows how Eq. 5 works and also shows the likely futility of the end-game maneuver: even if the interceptor fails to destroy the target, the target has maneuvered over 60 m off its intended course, possibly missing its target.

These examples lead to some observations:

- For engagements at hypersonic speeds, 50 g accelerations do little to change the situation even in the end-game of $t_{go,3} = 1$s. For all but the second example, the time to intercept changed little.
- In the second example, a tail-chase at identical speeds, the intercept would not have happened even without the maneuver and, owing to the hypersonic speeds compared with the available acceleration.
- Inspection of the Eqs. 2 and 3 does not reveal a clear rule-of-thumb for the velocity ratio. In the cases offered here, a ratio of 1.5 seems to do the job in all cases except a tail chase.
- A "velocity matching time" can characterize the interception regime. Defining $t_{match} = v_{c,1}/a_1'$ gives the time in which the target can deviate at a velocity comparable to the closing velocity. If t_{match}...

The generalized guidance equation does not provide a control law – it gives the t_{go} for an input $\Delta\vec{v}_1$. Allowing different response times for the target and interceptor[2] would provide a valuable extension to the generalized guidance equation. Also, the guidance law provides the basis for predictive guidance by setting the response time for the interceptor longer than that of the target. Finally, applying the guidance law iteratively with the target following an acceleration algorithm would provide a complete simulation of an engagement.

1 Appendix

Initial state at t_1 - on an intercept course. t_{go} is the time until intercept. Target begins a maneuver, accelerating at constant $\vec{a}_1'$. If no changes occur, the intercept will happen at $t_{intercept}$.

$$
\begin{aligned}
\vec{R}_1 &= \vec{r}_1' - \vec{r}_1 && (5)\\
\dot{\vec{R}}_1 &= \dot{\vec{r}}_1' - \dot{\vec{r}}_1 = \vec{u}_1 - \vec{v}_1 = \vec{v}_{c,1} && (6)\\
\vec{v}_{c,1} &= \vec{v}_1 - \vec{u}_1 && (7)\\
\vec{R}_1 &= \vec{v}_{c,1} t_{go} && (8)\\
\vec{R}_1 \cdot \vec{v}_{c,1} &> 0 && (9)\\
t_{intercept} &= t_1 + t_{go} && (10)
\end{aligned}
$$

[2]This change amounts to changing τ to τ' in the equations relating to the target and rederiving the guidance law.

At $t_2 = t_1 + \tau$, the target stops accelerating. The interceptor has measured the target's acceleration, computed the needed acceleration to restore intercept condition by t_3, and begun accelerating with $\vec{a}_1$.

$$
\begin{aligned}
\Delta\vec{u}_1 &= \vec{a}_1'\tau && (11)\\
\vec{r}_2' &= \vec{r}_1 + \vec{u}_1\tau + \Delta\vec{u}_1\tau/2 && (12)\\
\vec{r}_2 &= \vec{r}_1 + \vec{v}\tau && (13)\\
\vec{u}_2 &= \vec{u}_1 + \Delta\vec{u}_1 && (14)\\
\vec{v}_2 &= \vec{v}_1 && (15)\\
\vec{R}_2 &= \vec{R}_1 + \vec{u}_1\tau + \Delta\vec{u}_1\tau/2 - \vec{v}_1\tau && (16)\\
&= \vec{R}_1 + \vec{v}_{c,1}\tau + \Delta\vec{u}_1\tau/2 && (17)\\
\vec{v}_{c,2} &= \vec{v}_{c,1} + \Delta\vec{u} && (18)
\end{aligned}
$$

At $t_3 = t_1 + 2\tau$ $\vec{a}_1$ stops. $\Delta\vec{v}_1 = \vec{a}_1\tau$.

$$
\begin{aligned}
\vec{r}_3' &= \vec{r}_2' + \vec{u}_2\tau = \vec{r}_1' + \frac{3}{2}\vec{u}_1\tau + 2\Delta\vec{u}_1\tau && (19)\\
\vec{r}_3 &= \vec{r}_2 + \vec{v}_1\tau + \frac{\Delta\vec{v}_1\tau}{2} && (20)\\
&= \vec{r}_1 + \frac{1}{2}\Delta\vec{v}_B + \frac{1}{2}\Delta\vec{u}_1\tau + 2\vec{v}_1\tau && (21)\\
\vec{R}_3 &= \vec{r}_3' - \vec{r}_3 && (22)\\
&= \vec{R}_2 + (\vec{u}_1 - \vec{v}_1)\tau + (\Delta\vec{u}_1 - \frac{\Delta\vec{v}_1}{2})\tau && (23)\\
\Delta\vec{v}_1 &= \Delta\vec{v}_A + \Delta\vec{v}_B = \Delta\vec{u}_1 + \Delta\vec{v}_B && (24)\\
\Delta\vec{u}_1 - \frac{\Delta\vec{v}_A + \Delta\vec{v}_B}{2} &= \Delta\vec{u}_1 - \frac{\Delta\vec{u}_1 + \Delta\vec{v}_B}{2} = \frac{\Delta\vec{u}_1 - \Delta\vec{v}_B}{2} && (25)\\
\vec{R}_3 &= \vec{R}_2 + \vec{v}_{c,1}\tau + \frac{\Delta\vec{u}_1 - \Delta\vec{v}_B}{2}\tau && (26)\\
&= \vec{R}_1 + \vec{v}_{c,1}\tau + \frac{\Delta\vec{u}_1}{2} + \vec{v}_{c,1}\tau + \frac{\Delta\vec{u}_1 - \Delta\vec{v}_B}{2}\tau && (27)\\
&= \vec{R}_1 + 2\vec{v}_{c,1}\tau + \Delta\vec{u}_1\tau - \frac{\Delta\vec{v}_B\tau}{2} && (28)\\
\vec{u}_3 &= \vec{u}_2' = \vec{u}_1 + \Delta\vec{u}_1 && (29)\\
\vec{v}_3 &= \vec{v}_1 + \Delta\vec{v}_1 = \vec{v}_1 + \Delta\vec{u}_1 + \Delta\vec{v}_B && (30)\\
\vec{v}_{c,3} &= \vec{v}_1 - \Delta\vec{u}_1 + \vec{v}_1 - \Delta\vec{v}_1 = \vec{v}_{c,1} + \Delta\vec{v}_B && (31)
\end{aligned}
$$

$$
\begin{aligned}
\vec{R}_3 \times \vec{v}_{c,3} &= -\vec{R}_1 \times \vec{v}_{c,1} && (32)\\
& -\vec{R}_1 \times \Delta\vec{v}_B && (33)\\
& +2\vec{v}_{c,1}\tau \times \vec{v}_{c,1} (= 0) && (34)\\
& +\vec{v}_{c,1}\tau \times \Delta\vec{v}_B && (35)\\
& +\Delta\vec{u}\tau \times \vec{v}_{c,1} && (36)\\
& -\Delta\vec{u}_1\tau \times \Delta\vec{v}_B && (37)\\
& -\frac{3}{2}\Delta\vec{v}_B\tau \times \vec{v}_{c,1} && (38)\\
& +\frac{1}{2}\Delta\vec{v}_B\tau \times \Delta\vec{v}_B (= 0) && (39)\\
& && (40)
\end{aligned}
$$

Collecting terms gives the general guidance law:

$$
\Delta\vec{v}_B \times \left(\vec{R}_1 + \Delta\vec{u}_1\tau - \frac{3}{2}\vec{v}_{c,1}\tau\right) - \Delta\vec{u}_1 \times \vec{v}_{c,1}\tau - \vec{R}_1 \times \vec{v}_{c,1} = 0 \qquad (41)
$$

Next, insert $\vec{v}_{c,1} = -\vec{R}_1/t_{go}$ to obtain the restricted guidance law,

$$
\vec{R}_1 \times \Delta\vec{u}_1 \frac{\tau}{t_{go}} - \Delta\vec{v}_B \times \left(\vec{R}_1 \left(1 - \frac{3\tau}{2t_{go}}\right) - \Delta\vec{u}_1\tau\right) = 0 \qquad (42)
$$

$$
\begin{aligned}
\alpha &= \left(\vec{R}_1 + \Delta\vec{u}_1 - \frac{3}{2}\vec{v}_{c,1}\tau\right)_x && (43)\\
\beta &= -\left(\vec{R}_1 + \Delta\vec{u}_1 - \frac{3}{2}\vec{v}_{c,1}\tau\right)_z && (44)\\
\gamma &= \vec{v}_{c,1} \times \left(\Delta u_1\tau + \vec{R}_1\right)_y && (45)
\end{aligned}
$$

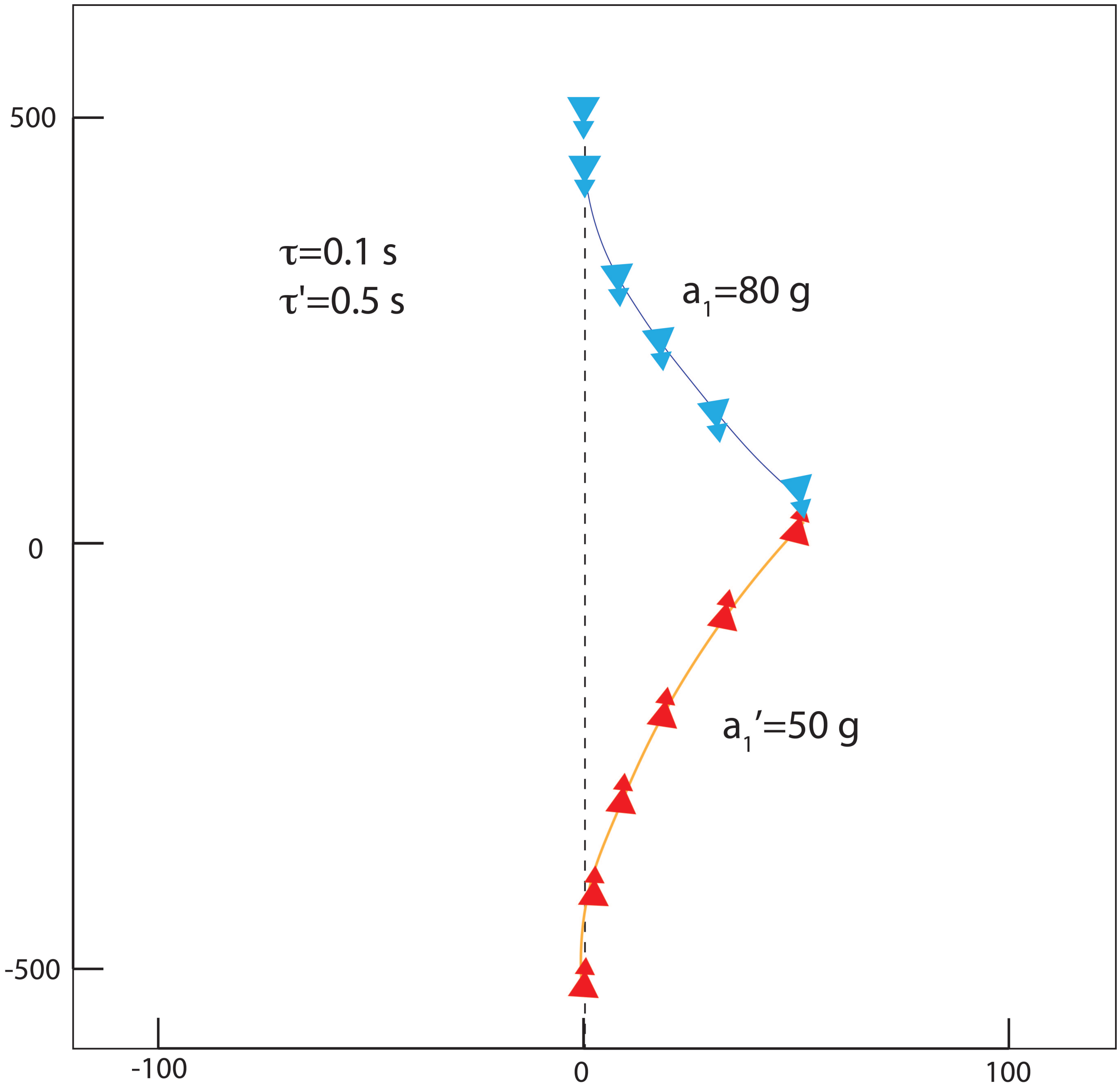

Figure 5: Final: $\tau = 0.1$s, $\tau' = 0.5$s, and $t_{go} = 0.5$s end-game intercept. End-game means the target has time for one last maneuver, $\tau' = t_{go}$. Vehicles are shown at 0.1 s intervals. At $t = 0$, the target accelerates along the x-axis with 50 g, and, 0.1s, the interceptor begins accelerating at 80g, as dictated by Eq. 4.